Study of A Particular Combinatorial Game

REU Summary Report

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1 Introduction

1.1 Combinatorial Game in General

Our research falls into the category of combinatorial game theory. Combinatorial game theory studies sequential games with perfect information, which means each player, when making any decision, is perfectly informed of all the events that have previously occurred. Impartial games are part of the combinatorial games in which allowable moves depend only on the positions and not on which of the two players is currently moving. The game that we are studying is impartial and involves two players. One of the most famous examples of impartial games is the game "Nim".

1.2 The "Nim" Game

"Nim" is a game in which there are arbitrary numbers of piles and each pile has arbitrary numbers of beans. In each move, the player must remove at least one bean and as many as the player wants as long as the beans come from the same pile. There are two kinds of winning condition for Nim. The first kind of condition stating that the player who takes the last bean wins is called "Normal Play", while the other condition stating that the player who takes the last bean loses and is called "Misere Play". Nim has been completely solved by mathematicians under both Normal Play and Misere Play.

1.3 The Sprague-Grundy Theorem

The Sprague-Grundy Theorem states that every impartial game under the normal play convention is equivalent to a Nim. Therefore, studying impartial games under Normal Play is no longer interesting to mathematicians. The game we have been studying is a Nim-like game, but under Misere Play. In general, Misere games are harder to solve compared to normal-played games and very few of them have been solved by mathematicians. The game that we have been studying is more complicated than any solved Misere games.

Note: There are two kinds of understandings for the notion of "Normal Play" and "Misere Play". Under first kind of understanding, "Normal Play" means you win if you take the last bean of the game, and "Misere Play" means you lose if you take the last bean of the game. Under second kind of understanding, "Normal Play" means you lose if you cannot move, and "Misere Play" means you win if you cannot move. These two ways of understanding are essentially the same, but we will utilize both of them later on because it is better to use different understandings in different contexts.

2 Our Game

2.1 Introduction and Terminology

General Information of our game:

- Settings: Two players, sequential play
- Objects: Beans
- Winning condition: Misere Play
- Allowable moves: Three kinds (specified in section 2.2)

We will introduce some terminologies to make future explanations clearer.

1. We are concerned with the structure of individual positions. Therefore, we also refer to an individual position in our combinatorial game by the term "game" which is often denoted as $\mathcal{G}$. We hope that the intended meanings can always be understood from the contexts.
2. The "size" of a particular pile refers to the number of beans in that pile.
3. The "size" of the game refers to the total number of beans in all piles.
4. The "pile number" or "multiplicity" of a certain pile size is the number of piles of that certain size in the game.
5. We state the winner or winning status of one game is $\mathcal{N}$ (next) if the player who moves next wins under optimal play and $\mathcal{P}$ (previous) if the player who moves next loses under optimal play.

2.2 Moves

In our game, we still have arbitrary number of piles while each pile has arbitrary number of beans. Unlike Nim, three kinds of moves are allowed in our game:

- Remove one pile of beans

- Remove one bean from a pile so that the size of the pile is reduced by 1
- Remove one bean from a pile so that this pile is broken down into two new piles and the sum of the size of two new piles is the original size minus 1

For example, the game that starts with one pile of three beans can be moved into the following games:

- No beans at all (move: $3^1 \to 0$)
- One pile of two beans (move: $3^1 \to 2^1$)
- Two piles each with one bean (move: $3^1 \to 1^2$)

2.3 Analysis

A family of games is a set of individual starting positions of this combinatorial game, usually with many parameters representing the multiplicities of different pile sizes. We use dots and superscripts to denote families of game. For example, the family of game that consists of i piles of size 2 and j piles of size 6 can be denoted as $2^i.6^j$ (i and j are parameters here). Sometimes we can even use 2.6 to denote this family of game since when the superscript of a pile size is absent we mean the multiplicities are parameters.

Our game is parameterized by $\mathbb{N}^r$ and we are trying to denote all of the possible moves by vectors $m_i = (k_1, k_2, ..., k_r) \in \mathbb{Z}^r$. Let $\mathcal{M}_\mathcal{G}$ be the set of all possible moves for a particular game $\mathcal{G}$ and $\mathcal{M}$ be the set of all possible moves for all games.

For example, let $r = 3$ and (n_1, n_2, n_3) denote the pile number of pile size $1, 2, 3$. The moves are:

$$3 \to 1,\ denoted\, as\, (1, 0, -1)$$
$$1^2 \to 0,\ denoted\, as\, (-2, 0, 0)$$

Note that each player has to remove at least one bean in every move, therefore the size of the game is always decreasing. We know $\mathcal{N}$ wins when the game size is reduced to 0. Therefore, we can always compute the results of games of any size in a recursive method and the winner is either $\mathcal{N}$ or $\mathcal{P}$.

$\mathbb{N}^r$ is the whole solution space of the game. We let $\mathbb{X}$ represent the set of $\mathcal{N}$ game, and $\mathbb{Y}$ represent the set of $\mathcal{P}$ game. Obviously, the whole solution space $\mathbb{N}^r$ is the disjoint union of $\mathcal{N}$ region $\mathbb{X}$ and $\mathcal{P}$ region $\mathbb{Y}$.

Finally, we would also like to point out that a game is $\mathcal{N}$ if and only if there exists a move after which the winner will be $\mathcal{P}$, and a game is $\mathcal{P}$ if and only if all the moves lead to a $\mathcal{N}$ game. This observation is very important when it comes to proving the empirical descriptions.

3 The Empirical Description

Since we are interested in the winners of the game of a given size, we can begin with some easy ones:

Let's write down some results for the game family 1^n as an example:

$$1^0\,\mathcal{N}$$
$$1^1\,\mathcal{P}$$
$$1^2\,\mathcal{N}$$
$$1^3\,\mathcal{P}$$
$$1^4\,\mathcal{N}$$
$$1^5\,\mathcal{P}$$
$$1^6\,\mathcal{N}$$

Naturally, we will have a guess based on the data up to size 6 for the game 1^n: The game is $\mathcal{N}$ if n is even; $\mathcal{P}$ if n is odd.

Then we can move on to the results of a little bit more complicated game family $1^n.2^m$:

$2^0\,\mathcal{N}$	$1.2^0\,\mathcal{P}$	$1^2.2^0\,\mathcal{N}$
$2^1\,\mathcal{N}$	$1.2^1\,\mathcal{N}$	$1^2.2^1\,\mathcal{N}$
$2^2\,\mathcal{P}$	$1.2^2\,\mathcal{N}$	$1^2.2^2\,\mathcal{P}$
$2^3\,\mathcal{N}$	$1.2^3\,\mathcal{N}$	$1^2.2^3\,\mathcal{N}$
$2^4\,\mathcal{P}$	$1.2^4\,\mathcal{N}$	$1^2.2^4\,\mathcal{P}$
$2^5\,\mathcal{N}$	$1.2^5\,\mathcal{N}$	$1^2.2^5\,\mathcal{N}$

Based on limited data, we can observe the following patterns:

(1)The results of the first column is the same as the results of the third column.

(2)The results of the first column is alternating after n is greater than 1.

(3)The results of the second column is the same after m is greater than 1.

Therefore, based on limited data, we can have a guess for the winning patterns of game family $1^n.2^m$:

- Guess:
  $1^n.2^m$ is $\mathcal{N} \iff$
  (1) n is even AND $m = 0$ or m is odd
  OR (2) n is odd and $m \geq 1$

This empirical guess turns out to be very easy to prove! We will prove this in Section 4.

$1^n.2^m$ is a game family parameterized by only 2 parameters n and m, but it has already revealed the only two elements of our descriptions for much more complicated game families: linear inequality conditions and parity conditions.

Therefore, we have a conjecture for any game family with r parameters:

- Conjecture:
  The $\mathcal{N}$ region and $\mathcal{P}$ region in $\mathbb{N}^r$ are the unions and complements of regions defined by finitely many linear inequalities and parity conditions.

Note: A linear equality condition can be considered as two linear inequality conditions, therefore linear equality conditions are also allowed to appear.

Since we have a conjecture now, we are going to describe all the game families using linear inequalities and parity conditions. We wrote a computer program to do that. The first step is always to divide a game family with r parameters into 2^r sub families of game, each with a fixed combination of parity. Then an algorithm is used to solve the sub families of game. The outline of our algorithm for solving a sub family of game that has a fixed combination of parity is as follows:

1. Assume we are studying $\mathcal{N}$ region. Find inequalities such that all points that fail the inequalities have the other winning status.
2. Determine how many boundaries we need to check on each inequality.
3. Check each boundary (when equalities hold in inequalities) and find the equalities and inequalities that further describe the set of points of the winning status.
4. If the description matches data, we are done. Otherwise, if the winning region we are studying is $\mathcal{N}$, return to step 1 and assume we are studying $\mathcal{P}$ region. If we are already studying $\mathcal{P}$ region and the description does not match the data, the algorithm fails.

The algorithm is implemented using C code, and details of the algorithm are omitted because of the complexity. However, this algorithm proved to be quite successful, on average more than 80% of the sub families of game can be correctly solved solely by the computer program. To get the description of the rest of the subgames, human efforts are needed. However, we are confident that we will be able get a complete empirical description of our game given sufficient time and efforts.

Now, in order to get a taste of what the description of the winning region looks like, we are going to examine the following example. The game family we are studying is 2.6.10.12.20.32. And the sub game family we are looking at is parity (0,0,0,0,0,0). The previous notation means the multiplicity of size 2, 6, 10, 12, 20 and 32 are all restricted to be even. The set of points we are going to describe is the set of points with winning status $\mathcal{P}$.

1. $mult(2) + mult(6) - mult(12) - mult(20) - mult(32) \geq 2$
2. $mult(2) + mult(10) - mult(12) - mult(20) - mult(32) \geq 2$
3. $mult(12) - mult(20) - mult(32) \leq 0$
4. $mult(2)+mult(20)-mult(12)-mult(32) > 0$ OR $(mult(2)+mult(20)-mult(12) - mult(32) = 0\, and\, mult(2) = 0)$

Here $mult(i)$ means the multiplicity of pile of size i. Any point satisfying the above four rules is a $\mathcal{P}$ point, and failure of any rule leads to a $\mathcal{N}$ position. For example, the point (2,2,2,0,0,0) which corresponds to the game $2^2.6^2.10^2$ is a $\mathcal{P}$ point because it satisfied all the above four rules, while the point (2,0,0,0,0,2) which corresponds to the game $2^2.32^2$ is a $\mathcal{N}$ point because it fails the first rule. Although the description is only an empirical one, we completely defined the set of $\mathcal{P}$ points when the parity is fixed. Thus, as conjectured, the winning region is defined by finitely many linear inequalities and parity conditions.

4 The Theoretical Proofs

One of the most amazing parts of our work is that we can check the equivalency of two infinite set by implementing finite check. We will prove the $\mathcal{N}$ and $\mathcal{P}$ regions for game family 1.2.3.4 to give a taste of what the final proving work will be like.

4.1 Our Conjecture

- Conjecture:
  The $\mathcal{N}$ region and $\mathcal{P}$ region in $\mathbb{N}^r$ are the unions and complements of regions defined by finitely many linear inequalities and parity conditions.

4.2 Our Goal

We have obtained our guess for the actual $\mathcal{N}$ region and $\mathcal{P}$ region through computer programs, and our goal is to prove our guess of the description of the $\mathcal{N}$ and $\mathcal{P}$ regions are actually correct.

In other words, for a fixed number r , we are trying to figure out whether our guess for the solution space is actually the same as the real solution space:

$$\begin{aligned} \mathbb{N}^r &= W_{actual} \bigsqcup L_{acutal} \\ &\stackrel{?}{=} W_{guess} \bigsqcup L_{guess} \end{aligned}$$

Note: $\bigsqcup$ is "disjoint union."

4.3 Lemma

Starting from Section 4.3, the move m_i begins to have two kinds of meanings. One is vector, the other one is map. For the sake of understanding, we will begin to use the map meaning from now on. For example, $m(\mathcal{G})$ is the image of game $\mathcal{G}$ under move $m \in \mathcal{M}_{\mathcal{G}}$, which is another game.

For the actual $\mathcal{N}$ and $\mathcal{P}$ region W_{actual} and L_{actual}, we have the following properties:

(1)$0 \in W_{actual}$

(2)$\forall x \in W_{actual}, \exists m \in \mathcal{M}_x \, s.t. \, m(x) \in L_{actual}$

(3)$\forall x \in L_{actual}, \forall m \in \mathcal{M}_x \, s.t. \, m(x) \in W_{actual}$

The intuition for the following properties is very simple:

(1) Point 0 is always in the $\mathcal{N}$ region because player $\mathcal{N}$ cannot move if there is no bean at all.

(2) Since we are playing optimally, a point x is in the $\mathcal{N}$ region if there exists one move to the $\mathcal{P}$ region. (It is enough if there is only one path to winning!)

(3) Still, since we are playing optimally, a point y is in the $\mathcal{P}$ region if all moves lead to the $\mathcal{N}$ region. (If it is a $\mathcal{P}$ point, it means that player $\mathcal{N}$ will lose this game whatever player $\mathcal{N}$ does!)

Here is our lemma:

- Suppose

$$\mathbb{N}^r = \mathbb{X}' \bigsqcup \mathbb{Y}'$$

s.t.

$$\begin{array}{ll} (a) & 0 \in \mathbb{X}' \\ (b) & \forall x \in \mathbb{X}', \exists m_1 \in \mathcal{M}_x \, s.t. \, m_1(x) \in \mathbb{Y}' \\ (c) & \forall y \in \mathbb{Y}', \forall m_i \in \mathcal{M}_y \, if \, m_i(y) \in \mathbb{N}^r \, then \, m_i(y) \in \mathbb{X}' \end{array}$$

then obviously $\mathbb{X} = \mathbb{X}'$ and $\mathbb{Y} = \mathbb{Y}'$

Our lemma is basically stating that we can finish the proof by checking the above three properties.

4.4 Proof of the Solution Space of Game Family 1.2.3.4

Game family 1.2.3.4 is the notation for the game family that only contains pile sizes 1~4. In other words, this kind of game has 4 free parameters n_1, n_2, n_3, n_4 to indicate the changing parameters of pile sizes 1~4. This kind of games can also be denoted as $1^{n_1}.2^{n_2}.3^{n_3}.4^{n_4}$

After empirically observed the results of the game up to size 173, we decided to break down our guess of the $\mathcal{N}$ region $\mathbb{X}'$ and $\mathcal{P}$ region $\mathbb{Y}'$ into many different sub regions X_i' and Y_i' according to the parity of the multiplicity of the pile sizes:

$$\mathbb{X}' = \bigsqcup_i X_i'$$
$$\mathbb{Y}' = \bigsqcup_i Y_i'$$

According to the lemma, this is enough to check:

$$\begin{array}{ll} (a) & 0 \in \mathbb{X}' \\ (b) & \forall i,\, X_i' \subseteq \bigcup_{j,k} m_k^{-1}(Y_j') \\ (c) & \forall i,\, m_k(Y_j') \subseteq \bigcup_{j,k} X_i' \end{array}$$

Note:

1.Since we have already considered point 0 in (a), we will not consider point 0 in (b). It will actually show contradiction if we consider point 0 in (b). The intuition here is that we do not have any moves when there is no beans, and therefore the game cannot be moved into any $\mathcal{P}$ regions, which means point 0 cannot be contained in any $m^{-1}(Y_i)$.

Thus, we are actually checking

$$\forall i,\, X_i' \setminus \{0\} \subseteq \bigcup_{j,k} m_k^{-1}(Y_j')$$

for (b).

Since there are already three proven rules: $1^2 \to 0$, $3 \to 1$ and $4 \to 2$ according to Prof.Snowden's previous work, we can reduce the game family 1.2.3.4 to $1^{\leq 1}.2$.

We are going to prove the validity of descriptions for $\mathcal{N}$ region and $\mathcal{P}$ region for the game family $G = 1^n.2^m$ with $n \leq 1$.

We broke this game family down into four different sub game families based on the parity of parameters, and obtained our guess for the descriptions X_i' and Y_i' from computer programs:

$$\begin{aligned} G &= \bigsqcup_i G_i \\ G_1 = 2^{2n}, &\quad X_1' = \{n = 0\}, \quad Y_1' = \{n \geq 1\} \\ G_2 = 2^{2n+1}, &\quad X_2' = \{n \geq 0\}, \quad Y_2' = \emptyset \\ G_3 = 1.2^{2n}, &\quad X_3' = \{n \geq 1\}, \quad Y_3' = \{n = 0\} \\ G_4 = 1.2^{2n+1}, &\quad X_4' = \{n \geq 0\}, \quad Y_4' = \emptyset \end{aligned}$$

Note: the computer program did not exclude point 0 when getting the descriptions, so we need to manually exclude it when doing the proofs.

All of the possible moves between these subgroups are listed:

$$m_1 : \underset{n \to n-1}{G_1 \to G_2} \qquad \textit{actual move} : 2^1 \to 0$$
$$m_2 : \underset{n \to n-1}{G_1 \to G_4} \qquad \textit{actual move} : 2^1 \to 1^1$$
$$m_3 : \underset{n \to n}{G_2 \to G_1} \qquad \textit{actual move} : 2^1 \to 0$$
$$m_4 : \underset{n \to n}{G_2 \to G_3} \qquad \textit{actual move} : 2^1 \to 1^1$$
$$m_5 : \underset{n \to n}{G_3 \to G_1} \qquad \textit{actual move} : 1^1 \to 0$$
$$m_6 : \underset{n \to n-1}{G_3 \to G_2} \qquad \textit{actual move} : 2^1 \to 1^1$$
$$m_7 : \underset{n \to n-1}{G_3 \to G_4} \qquad \textit{actual move} : 2^1 \to 0$$
$$m_8 : \underset{n \to n}{G_4 \to G_1} \qquad \textit{actual move} : 2^1 \to 1^1$$
$$m_9 : \underset{n \to n}{G_4 \to G_2} \qquad \textit{actual move} : 1^1 \to 0$$
$$m_{10} : \underset{n \to n}{G_4 \to G_3} \qquad \textit{actual move} : 2^1 \to 0$$

Note:

1. For all moves that map from n to $n-1$, the default domain of n is $n \geq 1$ because the multiplicity of the pile sizes have to be positive, or there will not even exist a move like this.

2. There is no maps like $G_1 \to G_3$ or $G_2 \to G_4$ because there is no moves like adding a pile of 1.

3. For m_6 and m_8, we used the rule $1^2 \to 0$ for simplification.

Part I

Obviously, $0 \in X'_1 \subseteq \mathbb{X}'$.

Part II

W.T.S.

$$\forall i,\ X'_i \subseteq \bigcup_{\substack{\forall m: G_i \to G_j \\ j \neq i}} m^{-1}(Y'_j)$$

Note: m's are all of the 10 kinds of moves listed above.

(1) $X'_1 = \{n = 0\}$

(2) $X'_2 = \{n \geq 0\}$

$$m_3^{-1}(Y'_1) = m_3^{-1}(n \geq 1) = \{n \geq 1\}$$
$$m_4^{-1}(Y'_3) = m_3^{-1}(n = 0) = \{n = 0\}$$

Therefore, $X'_2 = \{n \geq 0\} \subseteq \{n \geq 1\} \cup \{n = 0\} = m_3^{-1}(Y'_1) \cup m_4^{-1}(Y'_3)$

(3) $X_3' = \{n \geq 1\}$

$$m_5^{-1}(Y_1') = m_5^{-1}(n \geq 1) = \{n \geq 1\}$$
$$m_6^{-1}(Y_2') = m_6^{-1}(\emptyset) = \emptyset$$
$$m_7^{-1}(Y_4') = m_7^{-1}(\emptyset) = \emptyset$$

Therefore, $X_3' = \{n \geq 1\} \subseteq \{n \geq 1\} = m_5^{-1}(Y_1') \cup m_6^{-1}(Y_2') \cup m_7^{-1}(Y_4')$

(4) $X_4' = \{n \geq 0\}$

$$m_8^{-1}(Y_1') = m_8^{-1}(n \geq 1) = \{n \geq 1\}$$
$$m_9^{-1}(Y_2') = m_9^{-1}(\emptyset) = \emptyset$$
$$m_{10}^{-1}(Y_3') = m_{10}^{-1}(n = 0) = \{n = 0\}$$

Therefore, $X_4' = \{n \geq 0\} \subseteq \{n \geq 1\} \cup \{n = 0\} = m_8^{-1}(Y_1') \cup m_9^{-1}(Y_2') \cup m_{10}^{-1}(Y_4')$

In conclusion, $\forall i, j = 1 \sim 4 \, and \, j \neq i$, $X_i' \subseteq \bigcup_{\forall m: G_i \to G_j} m^{-1}(Y_j')$

Part III

W.T.S

$$\forall i, j = 1 \sim 4 \, and \, j \neq i, \quad \bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$$

(1) $Y_1' = \{n \geq 1\}$

$$m_1(Y_1') = m_1(n \geq 1) = \{n \geq 0\} \subseteq X_2'$$
$$m_2(Y_1') = m_2(n \geq 1) = \{n \geq 0\} \subseteq X_4'$$

Therefore, $i = 1, \forall j = 1 \sim 4 \, and \, j \neq i$, $\bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$

(2) $Y_2' = \emptyset$

Since the image of $\emptyset$ is always $\emptyset$ and $\emptyset$is always contained in any another set, $i = 2, \forall j = 1 \sim 4 \, and \, j \neq i$, $\bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$

(3) $Y_3' = \{n = 0\}$

$$m_5(Y_3') = m_5(n = 0) = \{n = 0\} \subseteq X_1'$$

Therefore, $i = 3, \forall j = 1 \sim 4 \, and \, j \neq i$, $\bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$

Note: moves m_6 and m_7 don't exist here because $n \geq 1$ is required for these two moves.

(4) $Y_4' = \emptyset$

Since the image of $\emptyset$ is always $\emptyset$ and $\emptyset$ is always contained in any another set, $i = 3, \forall j = 1 \sim 4\, and\, j \neq i, \bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$

In conclusion, $\forall i, j = 1 \sim 4\, and\, j \neq i, \bigcup_{\forall m: G_i \to G_j} m(Y_j') \subseteq \bigcup X_i'$

Finally, combing Part I~III, we have proven that our guess of the description is indeed the real $\mathcal{N}$ and $\mathcal{P}$ regions.

5 Ongoing Work

At present, we are trying to get our empirical description of game 1.2.3.4.5.6.7.8.

6 References

[1] Aaron N. Siegel. Misere Games and Misere Quotients. Lecture notes (arXiv:math/0612616v2), December 2006.