The Scaling Function Connection Problem in the Two-Dimensional Ising Model

William Warner
Mentor: Thomas Bothner

August 17, 2018

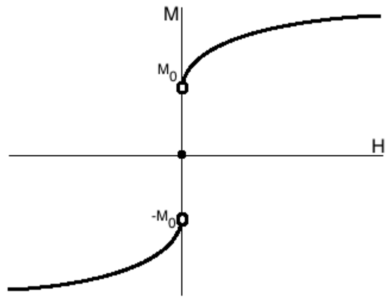
Abstract

We solve the connection problem associated with a ν-generalized function τ that appears in the massive scaling limit of the spin-spin correlation functions for the two-dimensional Ising model. In 1991 Craig Tracy solved this problem for the non-generalized case where $\nu = 0$ using operator theoretical methods [**?**]. Our method differs in that it relies purely on using the known short distance asymptotics of the underlying Painlevé-III transcendent along with representing the function τ in terms of a Hamiltonian dynamical system and its corresponding action integral. This method offers a simplified derivation for Tracy's 1991 result as well as a solution to the standing problem of calculating the connection coefficient in the short-distance asymptotics of τ for the ν-generalized case.

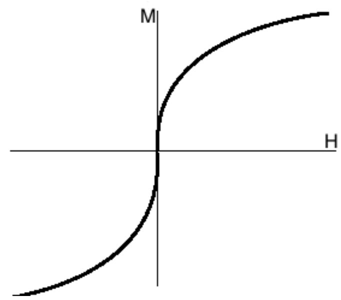
1 Introduction

The Ising model was originally introduced by physicists Wilhelm Lenz and Ernst Ising in the early 1920s as a model of ferromagnetism. On the microscopic level, single electrons in a material have an intrinsic magnetic dipole moment caused by the quantum mechanical physical property called "spin". The spin of an electron can take on two possible states, up or down. When spins of the electrons in a material become aligned, i.e. most or all are either spin up or spin down, the small magnetic contributions of each electron combine to form a measurable macroscopic magnetic field and we say that the material is magnetized. It is important to note that due to the Pauli Exclusion principle two electrons are able to occupy the same energy state in an atom by pairing up such that each energy state can have one spin up electron and one spin down electron. Due to this, the only materials that can become magnetized are those which have partially filled energy states. Thus, when we talk about the alignment of electron spins we most precisely mean the alignment of unpaired or "free" electron spins in a material.

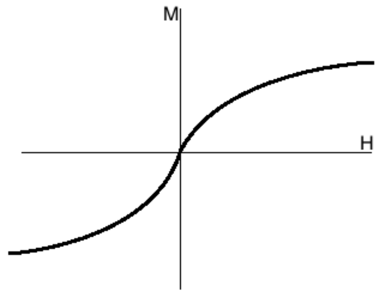
With that being said, one way for electron spins to become aligned is for them to be acted on by an external magnetic field. If we were to use an external magnetic field to magnetize a material and then lower that external magnetic field to zero one may expect the material to become unmagnetized. In ferromagnets this is not necessarily the case where under certain circumstances it is energetically favorable for free electrons to align their spins despite the absence of an external magnetic field. This process can only occur when the system is below



(a) $T < T_c$



(b) $T = T_c$



(c) $T > T_c$

Figure 1: Typical graphs of magnetization (M) vs external magnetic field (H) illustrating the phase transition at $T = T_c$ and the existence of spontaneous magnetization $\lim_{H \downarrow 0} M(H) = M_0 > 0$ for $T < T_c$.

some critical temperature, $T_c > 0$, known in physics as the Curie temperature. This physical phenomenon is known as spontaneous magnetization and represents a phase transition, i.e. an abrupt change of physical properties of the system, at the temperature T_c as shown in Figure 1.

One of the motivating questions for the Ising model is if it can predict this phase transition despite making no direct use of quantum mechanics. In 1925 Ising showed that in the one-dimensional Ising model no phase transition takes place, casting doubt on whether the model exhibits a phase transition for any dimension. But, in 1936 Rudolf Peierls reignited interest in the model by showing that a phase transition exists in the two-dimensional case. In 1941 Kramers and Wannier followed up this breakthrough by exactly determining the temperature $T_c > 0$ at which this phase transition takes place.

1.1 Two-Dimensional Ising Model

States of the Ising model consist of random, interacting spins $\sigma = \pm 1$, representing the magnetic dipole moments of electrons, at each grid point on the d-dimensional rectangular lattice $\mathbb{Z}^d$. To study the two-dimensional model we first analyze the finite rectangular lattice $\Lambda = \{(i, j) \in \mathbb{Z} \mid 0 \leq i \leq M, 0 \leq j \leq N\}$ with spins $\sigma_{ij} = \pm 1$ located at each grid point and pass afterwards to the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^2$ in some well-defined way. A key rule for the model is that spins only interact with their nearest neighbors. Thus, the general interaction energy for a given configuration $\sigma = (\sigma_{ij})$ on Λ has the form

$$\mathcal{E}_\Lambda(\sigma) = -J_1 \sum_{(i,j)\in\Lambda} \sigma_{ij}\sigma_{i\,j+1} - J_2 \sum_{(i,j)\in\Lambda} \sigma_{ij}\sigma_{i+1\,j}, \quad J_i > 0 \tag{1}$$

where J_1 represents the interaction between horizontal neighbors and J_2 between vertical neighbors. The simplified case where $J_1 = J_2$ is called the isotropic case but in general we will assume that J_1 and J_2 are not necessarily equal. There are many options for how to impose boundary conditions but in general we will use the periodic boundary conditions unless specifically stated otherwise. Periodic boundary conditions on the finite lattice are described by the relations: $\sigma_{i+M+1\,j} = \sigma_{ij}$ and $\sigma_{i\,j+N+1} = \sigma_{ij}$.

Following the general philosophy of Gibb's statistical mechanics, each configuration is assigned a probability depending on its interaction energy and the temperature of the system. For a given finite lattice Λ and configuration σ the configuration probability is given by the Gibbs measure

$$\mu_\Lambda(\sigma) = \frac{e^{-\beta\mathcal{E}_\Lambda(\sigma)}}{Z_\Lambda} \tag{2}$$

where $Z_\Lambda = \sum_\sigma e^{-\beta\mathcal{E}_\Lambda(\sigma)}$ (the partition function) acts as a normalization constant and $\beta = \frac{1}{k_B T} > 0$ is the inverse temperature of the system in terms of the Boltzmann constant k_B. Notice that the presence of the minus signs in (**??**) and (**??**) ensure that configurations with aligned spins are preferred.

The central physical interest lies on the computation of the Ising correlations which are simply the expected values of products of spins in the finite lattice. Concretely, for a finite set $A \subset \Lambda$ the Ising correlations are given by,

$$\left\langle \prod_{(i,j)\in A} \sigma_{ij} \right\rangle = \sum_\sigma \prod_{(i,j)\in A} \sigma_{ij}\mu_\Lambda(\sigma).$$

As stated earlier, the two-dimensional model (as well as higher dimensional ones) exhibits a phase transition at a critical temperature $T_c > 0$. The existence of this phase transition can be shown by calculating the expected value of the spin at the origin, σ_{00}, with a positive boundary condition (i.e. all boundary spins equal $+1$ simulating a weak external magnetic field) as $\Lambda \uparrow \mathbb{Z}^2$. This average is denoted by $\langle\cdot\rangle^+$ and we have, cf. [**?**]

$$\lim_{\Lambda\uparrow\mathbb{Z}^2} \langle\sigma_{00}\rangle^+ > 0 \text{ for } T < T_c \quad \text{and} \quad \lim_{\Lambda\uparrow\mathbb{Z}^2} \langle\sigma_{00}\rangle^+ = 0 \text{ for } T > T_c$$

where T_c (see Kramers, Wannier [**?**]) is determined by

$$\sinh(2\beta_c J_1)\sinh(2\beta_c J_2) = 1, \quad \beta_c = \beta(T_c) > 0.$$

This means that as T increases past T_c the boundary conditions go from having a nonzero effect on the state of the infinite system to having no effect at all. This abrupt change signals the existence of a phase transition. Moreover, the connection between this example and the observed physical phenomenon of spontaneous magnetization also follows. To make this connection, think of the positive boundary condition as a positive magnetic field acting on a ferromagnet. Much like how a magnetic field aligns the spins inside a metal, this positive boundary condition aligns the spins to mostly be $+1$ making their average to be greater than zero in the finite lattice. The process of taking the limit $\Lambda \uparrow \mathbb{Z}^2$ is analogous to removing the magnetic field from the system. The fact that any arbitrary spin (note that we can simply set the origin wherever we want before taking this limit) still has a slight positive alignment when below some critical temperature despite a magnetic field of zero is precisely the phenomenon of spontaneous magnetization.

Our interest in the two-dimensional Ising model focuses on the behavior of the two-point function $\langle \sigma_{00}\sigma_{mn}\rangle_\infty$, where $\langle\cdot\rangle_\infty$ represents the expected value taken after $\Lambda \uparrow \mathbb{Z}^2$, for an arbitrary $m, n \in \mathbb{Z}$ in the massive scaling limit specified below near the critical temperature, T_c.

1.2 Scaling Theory of $\langle \sigma_{00}\sigma_{mn}\rangle_\infty$

First we denote $R > 0$ to be the spatial distance between the spins σ_{00} and σ_{mn}. In the isotropic case R takes the familiar form $R = \sqrt{m^2 + n^2}$. But, in the more general case

$$R = \left(\sqrt{\frac{z_1(1-z_2^2)}{z_2(1-z_1^2)}}m^2 + \sqrt{\frac{z_2(1-z_1^2)}{z_1(1-z_2^2)}}n^2\right)^{\frac{1}{2}}$$

where $z_1 = \tanh(\beta J_1) \in (0,1)$ and $z_2 = \tanh(\beta J_2) \in (0,1)$ which can be thought of as a weighted spatial distance depending on the interaction energies J_1 and J_2. In [?] Wu, McCoy, Tracy, and Barouch showed that if we let $T \to T_c$ and $R \to \infty$ such that

$$\lim_{\substack{T\to T_c\\ R\to\infty}} \frac{|z_1z_2 + z_1 + z_2 - 1|}{\big(z_1z_2(1-z_2^2)(1-z_2^2)\big)^{\frac{1}{4}}} R = t$$

exists with $t \in (0,\infty)$, then

$$\lim_{\substack{T\uparrow\downarrow T_c\\ R\to\infty}} R^{\frac{1}{4}}\langle \sigma_{00}\sigma_{mn}\rangle_\infty = (2t)^{\frac{1}{4}}\big(\sinh(2\beta_c J_1) + \sinh(2\beta_c J_2)\big)^{\frac{1}{8}}\tau_\pm\left(t, \frac{1}{\pi}\right)$$

where

$$\tau_\pm(t,\lambda) = \exp\left[\frac{1}{4}\int_t^\infty \Big\{\sinh^2\psi(s,\lambda) - \Big(\frac{\mathrm{d}\psi}{\mathrm{d}s}(s,\lambda)\Big)^2\Big\} s\mathrm{d}s\right] \times \begin{cases} \sinh\frac{1}{2}\psi(s,\lambda) & (+) \\ \cosh\frac{1}{2}\psi(s,\lambda) & (-) \end{cases} \tag{3}$$

where $\lambda\pi \in [0,1]$ and ψ is a distinguished solution to the radial sinh-Gordon equation

$$\frac{d^2\psi}{dt^2} + \frac{1}{t}\frac{d\psi}{dt} = \frac{1}{2}\sinh(2\psi)$$

uniquely determined by the boundary condition

$$\psi(t,\lambda) \sim 2\lambda K_0(t), \quad t \to \infty$$

where $K_0(z)$ is the modified Bessel function, cf. [**?**].

Determining the asymptotic description of $\tau_\pm(t,\lambda)$ as $t \downarrow 0$ provided that the same description is given for $\tau_\pm(t,\lambda)$ as $t \to \infty$ (or vice versa) is known as the connection problem and is the problem of interest. This problem has been solved by Tracy [**?**] giving the results

$$\tau_\pm(t,\lambda) \sim \begin{cases} \lambda\sqrt{\frac{\pi}{2t}}e^{-t} & (+) \\ 1+\frac{\pi\lambda^2}{8t^2}e^{-2t} & (-) \end{cases}, \quad t\to\infty, \quad \lambda\pi\in[0,1]$$

$$\tau_\pm(t,\lambda) \sim A(\lambda)t^{\frac{\sigma}{4}(\sigma-2)}, \quad t\downarrow 0, \quad \lambda\pi\in(0,1], \quad \sigma=\frac{2}{\pi}\arcsin(\lambda\pi)\in(0,1]\,.$$

Computing $A(\lambda)$ and thus completing the description of $\tau_\pm(t,\lambda)$ as $t\downarrow 0$ is not a simple task. Tracy computed $A(\lambda)$ to be:

$$A(\lambda) = e^{3\xi'(-1)-(3s^2+\frac{1}{6})\ln 2}\big(G(1+s)G(1-s)\big)^{-1} \tag{4}$$

where $s=\frac{1}{2}(1-\sigma)$ and G is the Barnes-G function [**?**] defined in terms of the Gamma function Γ,

$$\ln G(1+z) = \frac{z}{2}\ln(2\pi) - \frac{z}{2}(z+1) + z\ln\Gamma(1+z) - \int_0^z \ln\Gamma(1+x)dx, \quad z\in\mathbb{C},\ \Re z > -1$$

Tracy did this by using operator theoretical methods, but in 2017 Bothner gave a simpler proof using Hamiltonian dynamical systems [**?**]. In this paper, we will show that Bothner's method can be pushed to solve an even more generalized case by introducing a certain parameter ν.

2 ν-Generalization of τ

The ν-generalization of τ provided by [**?**] is given below as:

$$\begin{aligned}\tau_\pm(t,\lambda,\nu) = \exp\Bigg[\frac{1}{4}\int_t^\infty \Big\{ \sinh^2\psi(s,\lambda,\nu) - \Big(\frac{\mathrm{d}\psi}{\mathrm{ds}}(s,\lambda,\nu)\Big)^2 \\ + \frac{4\nu}{s}\sinh^2\big(\frac{1}{2}\psi(s,\lambda,\nu)\big)\Big\} s\mathrm{d}s\Bigg] \times \begin{cases} \sinh\frac{1}{2}\psi(s,\lambda,\nu)\,(+) \\ \cosh\frac{1}{2}\psi(s,\lambda,\nu)\,(-) \end{cases}\end{aligned} \tag{5}$$

where $t>0$, $\lambda\pi\in[0,1]$, $\nu>-\frac{1}{2}$, and $\psi(t,\lambda,\nu)$ solves the ν-modified radial sinh-Gordon equation

$$\frac{d^2\psi}{dt^2} + \frac{1}{t}\frac{d\psi}{dt} = \frac{1}{2}\sinh(2\psi) + \frac{2\nu}{t}\sinh(\psi) \tag{6}$$

subject to the boundary condition

$$\psi(t,\lambda,\nu) \sim 2\lambda\int_1^\infty \frac{e^{-ty}}{\sqrt{y^2-1}}\left(\frac{y-1}{y+1}\right)^\nu dy, \quad t\to+\infty\,.$$

Notice that (??) is simply equal to (??) in the special case that $\nu = 0$.

Using results from [?] for the asymptotic expansion of the family of solutions to the Painlevé equation of the third kind we note that the asymptotic analysis of $\psi(t,\lambda,\nu)$ as $t \downarrow 0$ is

$$\psi(t,\lambda,\nu) = -\sigma \ln t - \ln B - \ln\left[1 - \frac{\nu}{B}(1-\sigma)^{-2}t^{1-\sigma} + B\nu(1+\sigma)^{-2}t^{1+\sigma} + \mathcal{O}\big(t^{2(1-\sigma)}\big)\right] \quad (7)$$

which holds for $\lambda\pi \in [0,1)$. For the point $\lambda = \frac{1}{\pi}$ we also have that

$$\psi(t,\frac{1}{\pi},\nu) = -\ln\left[\frac{t}{2}\left\{\nu \ln^2 t - C(\nu)\ln t + \frac{1}{4\nu}(C^2(\nu)-1)\right\}\right] + o(1) \quad (8)$$

where, for $\lambda\pi \in [0,1)$ and $\nu > \frac{1}{2}(\sigma - 1)$, the coefficients σ and B are functions of the parameters λ and ν

$$\sigma(\lambda) = \frac{2}{\pi}\arcsin(\lambda\pi), \quad B(\sigma,\nu) = 2^{-3\sigma}\frac{\Gamma^2\big(\frac{1}{2}(1-\sigma)\big)\Gamma\big(\nu+\frac{1}{2}(1+\sigma)\big)}{\Gamma^2\big(\frac{1}{2}(1+\sigma)\big)\Gamma\big(\nu+\frac{1}{2}(1-\sigma)\big)}$$

and $C(\nu) = 1 + 2\nu(3\ln 2 - 2\gamma_E - \psi_0(1+\nu))$. Here, γ_E is the Euler-Mascheroni constant and ψ_0 is the digamma function which will be further explained and used later in the paper.

With similar methods as in the non-generalized case (??), (??), and (??) can be substituted back into (??) to find the asymptotic descriptions

$$\tau_\pm(t,\lambda,\nu) \sim \begin{cases} \lambda\frac{\Gamma(\nu+\frac{1}{2})}{(2t)^{\nu+\frac{1}{2}}} & (+) \\ 1 - \frac{\lambda^2}{2}\frac{\Gamma^2(\nu+\frac{1}{2})}{(2t)^{2\nu+1}}e^{-2t}\Big\{\nu - \big(\nu+\frac{1}{2}\big)\big(\nu^2+\frac{3\nu}{2}+1\big)\frac{1}{t}\Big\} & (-) \end{cases} \qquad t \to \infty$$

and

$$\tau_\pm(t,\lambda,\nu) \sim A(\lambda,\nu)t^{\frac{\sigma}{4}(\sigma-2)}, \quad t\downarrow 0, \quad \lambda\pi \in (0,1], \quad \nu > \frac{1}{2}(\sigma-1)$$

Our central result of this paper is the calculation of $A(\lambda,\nu)$ through the manipulation of a Hamiltonian dynamical system.

Theorem 1. *For $\lambda\pi \in (0,1]$ and $\nu > \frac{1}{2}(\sigma-1)$*

$$A(\lambda,\nu) = e^{3\xi'(-1)-(3s^2+\frac{1}{6})\ln 2}\left(\frac{G(\nu+1-s)G(\nu+1+s)}{G^2(1-s)G^2(1+s)}\right)\frac{G^2(\frac{1}{2})\Gamma(\frac{1}{2})}{G^2(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \times \left(\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(\nu+1-s)\Gamma(\nu+s)}\right)^{\frac{\nu}{2}}$$

The proof of Theorem 1 will follow directly from our discussion and calculations presented in Section 2.1 below.

2.1 $A(\lambda,\nu)$ Calculation

Our goal is to obtain a closed form expression for $A(\lambda,\nu)$. The method we use involves partially rewriting the integrand of (**??**) as a Hamiltonian.

Proposition 1. *For $q=\psi$ and $p=-t\frac{d\psi}{dt}$,*

$$H=H(q,p,t,\nu)=\frac{t}{2}\sinh^2 q-\frac{p^2}{2t}+4\nu\sinh^2\frac{q}{2}$$

is a Hamiltonian function for the ν-modified radial sinh-Gordon equation given in (**??**)*.*

Proof. To prove that H is a Hamiltonian function for (**??**) we must check that (**??**) is equivalent to

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q}\,.$$

Using the assumption of the proposition and (**??**) we find

$$\frac{\partial H}{\partial p}=-\frac{p}{t}=\frac{d\psi}{dt}=\frac{dq}{dt}$$

and,

$$\begin{aligned}-\frac{\partial H}{\partial q}&=-t\sinh q\cosh q-4\nu\sinh\frac{q}{2}\cosh\frac{q}{2})\\&=-\frac{t}{2}\sinh(2q)-2\nu\sinh q\\&=-t\frac{d^2q}{dt^2}-\frac{dq}{dt}\\&=\frac{dp}{dt}\end{aligned}$$

□

We now utilize the classical action $S(t,\lambda,\nu)$ in our next proposition.

Proposition 2. *Let H, q, and p be defined as above and let $t>0$, $\lambda\pi\in[0,1]$, and $\nu>-\frac{1}{2}$. Then,*

$$\int_t^\infty H(q,p,s,\nu)ds=-tH(q,p,t,\nu)+S(t,\lambda,\nu)+4\nu\int_t^\infty\sinh^2\left(\frac{q}{2}\right)ds$$

where

$$S(t,\lambda,\nu)=\int_t^\infty\left(p\frac{dq}{ds}-H(q,p,s,\nu)\right)ds$$

Proof. By t-differentiating the right hand side we find

$$\begin{aligned}&\frac{d}{dt}\Big[-tH(q,p,t,\nu)+S(t,\lambda,\nu)+4\nu\int_t^\infty\sinh^2\frac{q}{2}ds\Big]=-H-t\frac{\partial H}{\partial t}-p\frac{dq}{dt}+H-d\nu\sinh^2\frac{q}{2}\\&=-t(\frac{t}{2}\sinh^2 q+\frac{p^2}{2t^2})+\frac{p^2}{t}-4\nu\sinh^2\frac{q}{2}\\&=-\frac{t}{2}\sinh^2 q+\frac{p^2}{2t}-4\nu\sinh^2\frac{q}{2}\\&=-H\end{aligned}$$

which of course matches what we would get by t-differentiating the left hand side. This means that both sides of the equation in Proposition 2 can only differ by a t-independent additive term. Both sides, however, decay exponentially fast as $t \to \infty$, implying that this additive term must be equal to zero. $\square$

We are able to further simplify $S(t, \lambda, \nu)$ by using integration by parts to switch integration of t to the λ parameter.

Proposition 3. *For any $t \in (0, \infty)$, $\lambda\pi \in [0, 1]$, and $\nu > -\frac{1}{2}$,*

$$S(t, \lambda, \nu) = -\int_0^\lambda p \frac{\partial q}{\partial \lambda'} d\lambda'$$

Proof. By λ-differentiating, using Proposition 1, and integrating by parts we find

$$\begin{aligned}
\frac{\partial S}{\partial \lambda} &= \int_t^\infty \left(\frac{\partial p}{\partial \lambda}\frac{\partial q}{\partial s} + p\frac{\partial}{\partial \lambda}\frac{\partial q}{\partial s} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial \lambda} - \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} \right) \\
&= p\frac{\partial q}{\partial \lambda} |_t^\infty - \int_t^\infty \frac{\partial p}{\partial s}\frac{\partial q}{\partial \lambda} ds + \int_t^\infty \frac{\partial p}{\partial s}\frac{\partial q}{\partial \lambda} ds \\
&= -p\frac{\partial q}{\partial \lambda} .
\end{aligned}$$

Therefore, the left and right hand sides can only differ by a λ-independent additive term. But, since $S = 0$ when $\lambda = 0$ it follows that this additive term is 0 and therefore Proposition 3 follows. $\square$

Summarizing, using Propositions 1 through 3 into (**??**) we derive the following formula for $\tau_\pm(t, \lambda, \nu)$:

$$\tau_\pm(t, \lambda, \nu) = \exp\left[-\frac{t}{2}H - \frac{1}{2}\int_0^\lambda p\frac{\partial q}{\partial \lambda'} d\lambda' + \nu\int_t^\infty \sinh^2\frac{q}{2} ds \right] \times \begin{cases} \sinh\frac{1}{2}q & (+) \\ \cosh\frac{1}{2}q & (-) \end{cases}$$

Computing $\int_t^\infty \sinh^2(\frac{q}{2})ds$ directly would be difficult, but luckily we can rewrite this into a simpler expression depending on p, $\frac{\partial q}{\partial \nu}$, and $\frac{\partial S}{\partial \nu}$.

Proposition 4. *Let $t > 0$, $\lambda\pi \in [0, 1]$, and $\nu > -\frac{1}{2}$. Then, using the definitions for p, q, and S, as above we have*

$$\int_t^\infty \sinh^2\frac{q}{2} ds = -\frac{1}{4}\left(p\frac{\partial q}{\partial \nu} + \frac{\partial S}{\partial \nu} \right)$$

Proof. Using Propositions 1 and 2 and integration by parts

$$\begin{aligned}
\frac{\partial S}{\partial \nu} &= \int_t^\infty \left[\left(\frac{\partial p}{\partial \nu}\frac{\partial q}{\partial s} + p\frac{\partial}{\partial s}\frac{\partial q}{\partial \nu} \right) - \left(\frac{\partial H}{\partial q}\frac{\partial q}{\partial \nu} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \nu} + \frac{\partial H}{\partial \nu} \right) \right] \quad ds \\
&= -p\frac{\partial q}{\partial \nu} - 4\int_t^\infty \sinh^2\left(\frac{q}{2}\right) ds
\end{aligned}$$

and the Proposition follows. $\square$

Now with Proposition 4, (**??**) becomes

$$\tau_{\pm}(t,\lambda,\nu) = \exp\left[-\frac{t}{2}H + \frac{1}{2}S - \frac{\nu}{4}\left(p\frac{\partial q}{\partial \nu} + \frac{\partial S}{\partial \nu}\right)\right] \times \begin{cases} \sinh\frac{1}{2}q & (+) \\ \cosh\frac{1}{2}q & (-) \end{cases}, \tag{9}$$
$$t > 0,\ \lambda\pi \in [0,1],\ \nu > -\frac{1}{2}.$$

Note that by t-differentiation of q we obtain

$$p = \sigma - \frac{\nu}{B}(1-\sigma)^{-1}t^{1-\sigma} + B\nu(1+\sigma)^{-1}t^{1+\sigma} + \mathcal{O}(t^{2-2\sigma}), \quad t \downarrow 0\,.$$

Plugging this formula of p and the above definition for $q = \psi$ as $t \downarrow 0$ into $-\frac{t}{2}H$ immediately gives

$$-\frac{t}{2}H = \frac{\sigma^2}{4} + \mathcal{O}(t^{1-\sigma}), \quad t \downarrow 0\,. \tag{10}$$

Now, using Proposition 3 for S we have

$$\begin{aligned} S &= \frac{\sigma^2}{2}\ln t + \int_0^\lambda \sigma \frac{\partial \ln B}{\partial \lambda'} d\lambda' + \mathcal{O}(t^{1-\sigma}\ln t) \\ &= \frac{\sigma^2}{2}\ln t - \frac{3}{2}\sigma^2 \ln 2 + 2\int_0^\lambda \sigma \frac{\partial}{\partial \lambda'} \ln \frac{\Gamma(\frac{1}{2}(1-\sigma))}{\Gamma(\frac{1}{2}(1+\sigma))} d\lambda' + \int_0^\lambda \sigma \frac{\partial}{\partial \lambda'} \ln \frac{\Gamma(\nu + \frac{1}{2}(1+\sigma))}{\Gamma(\nu+\frac{1}{2}(1-\sigma))} d\lambda' \\ &\qquad + \mathcal{O}(t^{1-\sigma}\ln t)\,. \end{aligned}$$

Introducing the definitions

$$s = \frac{1}{2}(1-\sigma), \quad J(\lambda,\nu) = \int_0^\lambda \sigma \frac{\partial}{\partial \lambda'} \ln \frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)} d\lambda'$$

S can be rewritten as

$$S = \frac{\sigma^2}{2}\ln t - \frac{3}{2}\sigma^2 \ln 2 - 2J(\lambda,0) + J(\lambda,\nu)\,. \tag{11}$$

To evaluate the integral $J(\lambda,\nu)$ we rely on the following identity for Γ and the Barnes-G function,

$$\int_0^z \ln\Gamma(1+x)dx = \frac{z}{2}\ln(2\pi) - \frac{z}{2}(z+1) + z\ln\Gamma(1+z) - \ln G(1+z), \quad z \in \mathbb{C}, \Re z > -1 \tag{12}$$

the functional equations,

$$\Gamma(1+z) = z\Gamma(z), \quad G(1+z) = \Gamma(z)G(z) \tag{13}$$

and the values,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad 2\ln G\left(\frac{1}{2}\right) = 3\xi'(-1) - \frac{1}{2}\ln\pi + \frac{1}{12}\ln 2 \tag{14}$$

which are all available on NIST [**?**].

Proposition 5. *For $\sigma \in [0,1)$ and $\nu > \frac{1}{2}(\sigma - 1)$,*

$$J(\lambda,\nu) = \frac{\sigma^2}{2} - (2\nu - 1)\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + 2\ln\frac{G(\nu+1-s)G(\nu+s)}{G^2(\nu+\frac{1}{2})}\,.$$

Proof. Starting with the above definition of $J(\lambda,\nu)$ and using integration by parts we have

$$\begin{aligned}
J(\lambda,\nu) &= \sigma\ln\frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)} - \int_0^\lambda \frac{\partial\sigma}{\partial\lambda'}\ln\frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)}d\lambda' \\
&= \sigma\ln\frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)} - \int_0^\lambda \frac{\partial\sigma}{\partial\lambda'}\ln\Gamma(\nu+1-s)d\lambda' + \int_0^\lambda \frac{\partial\sigma}{\partial\lambda'}\ln\Gamma(\nu+s)d\lambda'
\end{aligned}$$

and using the substitutions $u = \nu - s$ and $v = \nu + s - 1$

$$J(\lambda,\nu) = \sigma\ln\frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)} - 2\int_{\nu-\frac{1}{2}}^{\nu-s}\ln\Gamma(1+u)du - 2\int_{\nu-\frac{1}{2}}^{\nu+s-1}\ln\Gamma(1+v)dv\,.$$

Now, with (**??**) and (**??**) and algebraic manipulations we find

$$J(\lambda,\nu) = \frac{\sigma^2}{2} - (2\nu - 1)\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + 2\ln\frac{G(\nu+1-s)G(\nu+s)}{G^2(\nu+\frac{1}{2})}\,.$$

□

Using this derivation for $J(\lambda,\nu)$ along with the use of the special values in (**??**) we derive the following expression for S:

$$\begin{aligned}
S(\lambda,\nu) = \frac{\sigma^2}{2}\ln t + \left(\frac{1}{6} - \frac{3}{2}\sigma^2\right)\ln 2 - \frac{\sigma^2}{2} + 6\xi'(-1) + 2\ln\left[\frac{G(\nu+1-s)G(\nu+1+s)}{G^2(1-s)G^2(1+s)}\right] \\
+ \ln\left[\frac{\Gamma(\nu+1-s)\Gamma^2(s)}{\Gamma(\nu+s)\Gamma^2(1-s)}\right] + 2\ln\left[\frac{G^2(\frac{1}{2})\Gamma(\frac{1}{2})}{G^2(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\right] \\
- 2\nu\ln\left[\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})}\right] + \mathcal{O}(t^{1-\sigma}\ln t)
\end{aligned} \tag{15}$$

Now all that's left is to evaluate the $p\frac{\partial q}{\partial\nu} + \frac{\partial S}{\partial\nu}$ term. In order to do so we will need to use the digamma function

$$\psi_0(x) = \frac{d}{dx}\ln\Gamma(x), \quad x \notin \mathbb{Z}_{\le 0}$$

along with the identity

$$\frac{d}{dx}\ln G(x) = (x-1)\psi_0(x) - x + \frac{1}{2}\ln(2\pi) + \frac{1}{2} \tag{16}$$

which follows from (**??**).

Proposition 6. *As $t \downarrow 0$ for $\sigma \in [0,1)$ and $\nu > \frac{1}{2}(\sigma - 1)$*

$$p\frac{\partial q}{\partial \nu} + \frac{\partial S}{\partial \nu} = -2\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + \mathcal{O}(t^{1-\sigma}\ln t)$$

Proof. For the first term we find

$$\begin{aligned} p\frac{\partial q}{\partial \nu} &= -\sigma\frac{\partial}{\partial \nu}\ln B + \mathcal{O}(t^{1-\sigma}) \\ &= -\sigma\big(\psi_0(\nu+1-s) - \psi_0(\nu+s)\big) + \mathcal{O}(t^{1-\sigma})\,. \end{aligned}$$

For the second term, from (**??**) we have

$$\begin{aligned} \frac{\partial S}{\partial \nu} &= 2\frac{\partial}{\partial \nu}\ln\big(G(\nu+1-s)G(\nu+1+s)\big) + \frac{\partial}{\partial \nu}\frac{\Gamma(\nu+1-s)}{\Gamma(\nu+s)} - 2\frac{\partial}{\partial \nu}\ln\big(G^2(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})\big) \\ &\quad - 2\frac{\partial}{\partial \nu}\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} - 2\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + \mathcal{O}(t^{1-\sigma}\ln t) \end{aligned}$$

and after applying (**??**) we find

$$\frac{\partial S}{\partial \nu} = \sigma\big(\psi_0(\nu+1-s) - \psi_0(\nu+s)\big) - 2\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + \mathcal{O}(t^{1-\sigma}\ln t)\,.$$

After combining the two terms we are left with

$$p\frac{\partial q}{\partial \nu} + \frac{\partial S}{\partial \nu} = -2\ln\frac{\Gamma(\nu+1-s)\Gamma(\nu+s)}{\Gamma^2(\nu+\frac{1}{2})} + \mathcal{O}(t^{1-\sigma}\ln t)\,.$$

$\square$

Now, since

$$\left\{\begin{array}{ll} \sinh\frac{1}{2}q & (+) \\ & \\ \cosh\frac{1}{2}q & (-) \end{array}\right. = \frac{1}{2}t^{-\frac{\sigma}{2}}B^{-\frac{1}{2}}\big(1+\mathcal{O}(t^{1-\sigma})\big), \quad t\downarrow 0$$

and combining (**??**), Proposition 6, and (**??**) back into (**??**) we obtain

$$\begin{aligned} \tau(t,\lambda,\nu) &= t^{\frac{\sigma}{4}(\sigma-2)}e^{3\xi'(-1)-(3s^2+\frac{1}{6})\ln 2}\left(\frac{G(\nu+1-s)G(\nu+1+s)}{G^2(1-s)G^2(1+s)}\right)\frac{G^2(\frac{1}{2})\Gamma(\frac{1}{2})}{G^2(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \\ &\quad\times\left(\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(\nu+1-s)\Gamma(\nu+s)}\right)^{\frac{\nu}{2}}\big(1+\mathcal{O}(\max\{t^{1-\sigma}\ln t, t^{\sigma}\})\big) \end{aligned}$$

thus proving Theorem 1. Notice that by setting $\nu = 0$ in the formula above and using the functional equations and special values for the Barnes-G and Gamma functions we return (**??**), the equation for $A(\lambda)$ in the non-generalized case.

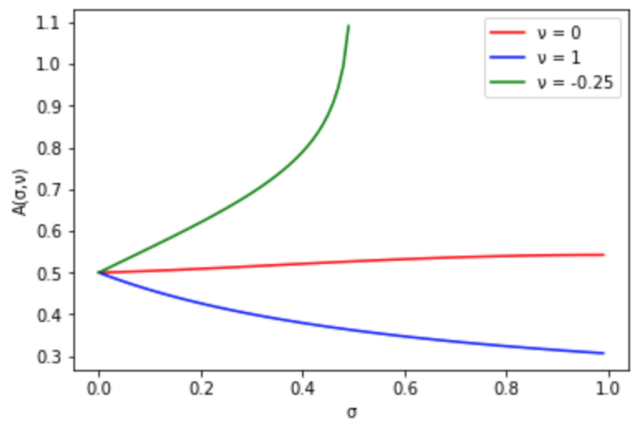


Figure 2: $A(\sigma, \nu)$ vs. σ for specific values of ν.

3 Future Work: Numerics

Ideally we would like to be able to computationally check our result for $A(\lambda, \nu)$. Using the computing language Python along with the library mpmath, which contains all the special functions needed, plotting $A(\lambda, \nu)$ for specific values of λ and ν is easy. For the simplicity of plotting A along the interval $[0, 1]$ we plot $A(\sigma, \nu)$ for certain values of ν seen in Figure 2.

To obtain $A(\lambda, \nu)$ through direct computation we would first need to directly compute the Painlevé-III transcendent $\psi(t, \lambda, \nu)$ as $t \downarrow 0$ and use this to computationally analyze $\tau_{\pm}(t, \lambda, \nu)$ as $t \downarrow 0$. One avenue of doing such a computation relies on the use of the Fredholm determinant which is a generalization of the determinant of a finite dimensional operator. The Fredholm determinant has the form

$$d(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_I \cdots \int_I \det[K(x_i, x_j)]_{i,j=1}^n dx_1 \ldots dx_n$$

where $K(x, y)$ is called the kernel and represents an associated integral operator and is a function $I \times I \subset \mathbb{R}^2$.

In 1977 McCoy, Tracy, and Wu [?] showed that ψ can be written as the infinite sum

$$\psi(t, \lambda, \nu) = \sum_{n=0}^{\infty} \lambda^{2n+1} \psi_{2n+1}(t, \nu)$$

where

$$\psi_1(t, \nu) = 2 \int_1^{\infty} \frac{e^{-ty}}{(y^2 - 1)^{\frac{1}{2}}} \left(\frac{y-1}{y+1} \right)^{\nu} dy$$

and for $n \geq 1$

$$\psi_{2n+1}(t,\nu) = \frac{2}{2n+1}\int_1^\infty \cdots \int_1^\infty \left[\prod_{j+1}^{2n+1} \frac{e^{-ty_j}}{y_j + y_{j+1}}\right] \times \left[\prod_{j+1}^{2n+1}\left(\frac{y_j-1}{y_j+1}\right)^{\nu-\frac{1}{2}} + \prod_{j+1}^{2n+1}\left(\frac{y_j-1}{y_j+1}\right)^{\nu+\frac{1}{2}}\right] dy_1 \ldots dy_{2n+1}\,.$$

In 2000 Harold Widom [?] showed that if we set the kernel to be

$$K(x,y) = \frac{e^{-t(x+x^{-1})/2}}{x+y}\left|\frac{x-1}{x+1}\right|^{2\nu}$$

on the interval $(0,\infty)$ then ψ can be represented as

$$\psi = \ln\left(\frac{d(\frac{\lambda}{2})}{d(\frac{-\lambda}{2})}\right).$$

In 2010, Folkemar Bornemann [?] provided a systematic way of computing Fredholm determinants by approximating it to $d_Q(z)$ defined as the determinant of the $m \times m$ matrix

$$d_Q(z) = \det\left(\delta_{ij} + z w_i^{\frac{1}{2}} K(x_i, x_j) w_j^{\frac{1}{2}}\right)_{i,j=1}^m$$

where the w_i and x_i represents the weights and sample points for some Gaussian quadrature rule and the integer m can be varied depending on the user's desired accuracy.

Initial attempts to apply this procedure to our problem, however, have failed due to the blowup of ψ as $t \downarrow 0$. This causes slow convergence and chaotic behavior as we attempt to compute ψ close to $t = 0$. We plan to address this issue as well as the subsequent numerical evaluation of $\tau_\pm(t,\lambda,\nu)$ in our future work.

References

[1] Bornemann, F.: On the Numerical Evaluation of Fredholm Determinants. Mathematics of Computation Volume 79, Number 270, 871-915 (2010)

[2] Bothner, T.: A Short Note on the Scaling Function Constant Problem in the Two-Dimensional Ising Model. J Stat Phys. **170**, 672-683 (2017)

[3] McCoy, B., Tracy, C., Wu, T.: Painlevé functions of the third kind. J. Math. Phys. **18**, 1058-1092 (1977)

[4] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov

[5] Palmer, J.: Planar Ising Correlations. Progress in Mathematical Physics, 49. Birkhäuser Boston Inc., Boston, MA. (2007)

[6] Tracy, C.: Asymptotics of a τ-function arising in the two-dimensional Ising model. Commun. Math. Phys. **142**, 297-311 (1991)

[7] Kramers, H., Wannier, G.: Statistics of the two-dimensional ferromagnet. Physical Review. **60**, 252-262 (1941)

[8] Widom, H.: On the Solution of a Painlevé III Equation. Mathematical Physics, Analysis and Geometry. **3**, 375-384 (2000)

[9] Wu, T., McCoy, B., Tracy, C., Barouch, E.: Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region. Phys. Rev. **13**, 316-374 (1976)