Minimal Triangulations of Sphere Products

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Abstract

We provide an alternative construction of a $(2d{+}2)$-vertex triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. We generalize this construction to give a triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.

1 Introduction

A basic question in the field of combinatorial geometry is asking for the existence of a vertex-minimal triangulation of a combinatorial manifold. There are few manifolds for which this is known, much less how to construct such a triangulation. For sphere products, Brehm and Kuhnel [1] prove that, for $i \leq j$, the vertex-minimal triangulation of $\mathbb{S}^i \times \mathbb{S}^j$ requires at least $i + 2j + 4$ vertices. Such triangulations are constructed by Lutz [5] in low-dimensional cases with the aid of computer programs. However, such constructions are not known to exist in general. More recently, Klee and Novik [3] found a centrally symmetric 2d-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ for all pairs of integers (i, d) with $0 \leq i \leq d-2$. Their construction can be realized as certain subcomplexes of the $(d-1)$-dimensional octahedral sphere.

In the interest of gathering information about the kinds of constructions that triangulate sphere products, in Section 3, we give an alternative construction of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$, and list some properties of this triangulation. The construction relies on the use of Klee and Novik's subcomplexes, outlined in Section 2 along with other basic information concerning simplicial complexes and combinatorial manifolds. In Section 4, we generalize this result to create triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-3}$ for all pairs of integers (i, d) such that $2 \leq i \leq d-3$.

2 Preliminaries

In this section, we review concepts and results concerning simplicial complexes and combinatorial manifolds, and state other necessary information.

A *simplicial complex* Δ on a vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, which is closed under inclusion and contains $\{v\}$ for all $v \in V$. For $\sigma \in \Delta$, define $\dim(\sigma) := |\sigma| - 1$, and define the *dimension* of Δ to be the maximal dimension of the faces of Δ. The *facets* of Δ are precisely the maximal faces of Δ with respect to inclusion. we say that Δ is *pure* if all facets of Δ have the same dimension.

Let σ be a face of a simplicial complex Δ. The *link* of σ in Δ, $\text{lk}_\Delta(\sigma)$, and *star* of σ in Δ, $\text{st}_\Delta(\sigma)$ are defined by

$$\text{lk}_\Delta(\sigma) := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\} \text{ and } \text{st}_\Delta(\sigma) := \{\tau \in \Delta : \tau \cup \sigma \in \Delta\}.$$

When it is clear which complex σ is in, we denote the link and star of σ simply as $\text{lk}(\sigma)$ and $\text{st}(\sigma)$, respectively.

Let Δ be a pure d-dimensional simplicial complex. For facets $\tau_1, \ldots, \tau_k \in \Delta$, we say that the ordering $(\tau_1, \ldots, \tau_k)$ is a *shelling* of Δ if for every $1 < i \leq k$, the set of faces $\tau_i - (\cup_{j<i}\tau_j)$ has a unique minimal element with respect to inclusion. We say that this minimal face is the *restriction* of τ_i, denoted $r(\tau_i)$. Equivalently, $(\tau_1, \ldots, \tau_k)$ is a shelling if for every $1 < i \leq k$, the complex $\Delta \cap (\cup_{j<i}\tau_j)$ is pure and has dimension $d - 2$.

A d-dimensional simplicial complex Δ is said to be a *combinatorial manifold* if the link of every non-empty face of Δ is a triangulated $(d-|\sigma|)$-dimensional ball or sphere. A combinatorial ball is a combinatorial manifold that triangulates a ball, and likewise, a combinatorial sphere is a combinatorial manifold that triangulates a sphere. The boundary complex of a simplicial d-ball is a simplicial $(d-1)$-sphere.

Throughout, we denote by ∂C_d^* the boundary complex of the d-dimensional cross-polytope on $2d$ vertices. let $\{x_1, x_2, \ldots, x_d, y_1, \ldots, y_d\}$ be the vertex set of ∂C_d^*, where x_i and y_i are antipodal vertices for all i. The facets of ∂C_d^* are precisely the faces of ∂C_d^* with exactly one element taken from each antipodal pair of vertices. Thus each facet of ∂C_d^* may be uniquely identified with an xy-word of length d, and ∂C_d^* is a subcomplex of ∂C_{d+1}^* by adding $\{x_{d+1}, y_{d+1}\}$ to the vertex set of ∂C_d^*.

For a facet $\tau = \{v_1, ..., v_d\}$ of ∂C_d^*, we say that τ has a *switch* at index i if v_i and v_{i+1} have differing labels. Define $\mathcal{B}(i,d)$ by the set of facets of ∂C_d^* with at most i switches. By definition, $\mathcal{B}(i,d)$ is a subcomplex of ∂C_d^*. Finally, the following lemma is taken directly from [3]. Here $\mathcal{D}_i$ denotes the dihedral group of order $2i$:

Lemma 2.1. *For $0 \leq i < d-1$, the complex $\mathcal{B}(i,d)$ satisfies the following:*

a) $\mathcal{B}(i,d)$ contains the entire i-skeleton of the d-dimensional cross polytope as a subcomplex.

b) $\mathcal{B}(i,d)$ is centrally symmetric. Moreover, it admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_d$ if i is even and of $\mathcal{D}_{2d}$ if i is odd.

c) The complex of $\mathcal{B}(i,d)$ in the boundary complex of the d-dimensional cross polytope is simplicially isomorphic to $\mathcal{B}(d-i-2,d)$.

d) $\mathcal{B}(i,d)$ is a combinatorial manifold (with boundary) whose integral (co)homology groups coincide with those of $\mathbb{S}^i$.

e) The boundary of $\mathcal{B}(i,d)$ is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.

3 Triangulations of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

In this section, we construct an alternative triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 5$. To do this, we will make use of the following theorem in [4].

Theorem 3.1. *Let M be a simply connected codimension-1 submanifold of $\mathbb{S}^d$, where $d \geq 5$. If M has the homology of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ and $1 < i \leq \frac{d-1}{2}$, then M is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$.*

Proposition 3.2. *Let D_1 and D_2 be two combinatorial d-balls such that $\partial(D_1 \cup D_2)$ is a $(d-1)$-dimensional submanifold of a combinatorial d-sphere and $D_1 \cap D_2 \subseteq \partial D_1 \cap \partial D_2$ is a path-connected combinatorial $(d-1)$-manifold that has the same homology as S^{i-1}. Then $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ for $d \geq 5$ and $1 < i \leq \frac{d-1}{2}$.*

Proof: First note that $D_1 \cup D_2$ is the union of two combinatorial d-balls that intersect along the combinatorial $(d-1)$-manifold $D_1 \cap D_2$. Hence $D_1 \cup D_2$ is a combinatorial d-manifold with boundary, and $\partial(D_1 \cup D_2)$ is a combinatorial $(d-1)$-manifold.

By applying the Mayer-Vietoris sequence on $(\partial D_1 \backslash \partial D_2, D_1 \cap D_2, \partial D_1)$, we see that the complex $\partial D_1 \backslash \partial D_2$ has the same homology as $\mathbb{S}^{d-i-1}$. Applying the Mayer-Vietoris sequence on the triple $(\partial D_1 \backslash \partial D_2, \partial D_2 \backslash \partial D_1, \partial(D_1 \cup D_2))$, we obtain that $\partial(D_1 \cup D_2)$ has the same homology as $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. On the other hand, the complex $D_1 \cup D_2$ is simply connected, since the union of two simply connected open subsets $\text{int}(D_1), \text{int}(D_2)$ with path-connected intersection $D_1 \cap D_2$ is simply connected. We conclude from Theorem 3.1 that $\partial(D_1 \cup D_2)$ triangulates $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. □

The above proposition provides us with a method of constructing a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

3.1 Preparations

Notation 3.3. In the following we use the convention that in ∂C_d^*, $x_{d+k} := x_k$ and $y_{d+k} := y_k$. In addition, fix $j = \left\lfloor \frac{d+1}{2} \right\rfloor$.

Definition 3.4. Let τ be a face of ∂C_d^* and let $\kappa(\tau)$ count the number of y labels in τ. Define Γ_k to be the union of facets τ in ∂C_d^* that have at most 2 switches and with $\kappa(\tau) = k$.

It is easy to see that for $1 \leq k \leq d-1$, the complex Γ_k consists of d facets

$$\tau_k^l = (\{x_1, \ldots, x_d\} \backslash \{x_l, \ldots x_{l+k-1}\}) \cup \{y_l, \ldots, y_{l+k-1}\}, \text{ where } 1 \leq l \leq d.$$

We will now prove some properties of Γ_k.

Proposition 3.5. *The complex $\cup_{k=0}^n \Gamma_k$ is a shellable $(d-1)$-ball for all $0 \leq n \leq j$.*

Proof: We prove by induction on n. For $n = 0$, it is easy to see that the complex Γ_0 is a shellable $(d-1)$-ball. Now assume that $\Delta := \cup_{k=0}^{n-1}\Gamma_k$ is a shellable $(d-1)$-ball. Note that for any $1 \leq k \leq d$, the restriction face of τ_n^k is $r(\tau_n^k) = \{y_k, y_{k+n-1}\}$, and $\tau_n^k \cap (\Delta \cup \cup_{m<k}\tau_n^m) = [r(\tau_n^k), \tau_n^k]$, which is a shellable $(d-2)$-ball. Hence by the inductive hypothesis and induction on n, $\cup_{k=0}^{n}\Gamma_k$ is a simplicial $(d-1)$-ball that has a shelling order $(\tau_0^1, \tau_1^1, \ldots, \tau_1^d, \ldots \tau_n^1, \ldots, \tau_n^d)$.
□

Proposition 3.6. *The complex $\Gamma_{j-1} \cup \Gamma_j$ has the same homology as $\mathbb{S}^1$.*

Proof: The complex $\Gamma_{j-1} \cup \Gamma_j$ consists of $2d$ facets $\tau_{j-1}^1, \ldots, \tau_{j-1}^d, \tau_j^1, \ldots, \tau_j^d$. Each facet τ_{j-1}^k has exactly two adjacent facets τ_j^k and τ_j^{k-1} in $\Gamma_{j-1} \cup \Gamma_j$. Similarly, each facet τ_j^k has exactly two adjacent facets τ_{j-1}^k and τ_{j-1}^{k+1} in $\Gamma_{j-1} \cup \Gamma_j$. Hence the facet ridge graph of $\Gamma_{j-1} \cup \Gamma_j$ is a $2d$-cycle. Thus $\Gamma_{j-1} \cup \Gamma_j$ has the same homology as S^1. □

Definition 3.7. For $1 \leq k \leq d-1$, define γ_k to be the set of facets in $\Gamma_{k-1} \cup \Gamma_{k+1}$ such that v_1 has the same label as v_d. For example, when $d = 6$, the following facets are in γ_i:

$$\{x_1, x_2, x_3, y_4, y_5, x_6\}, \{x_1, x_2, y_3, y_4, x_5, x_6\}, \{x_1, y_2, y_3, x_4, x_5, x_6\},$$
$$\{y_1, y_2, y_3, x_4, x_5, y_6\}, \{y_1, y_2, x_3, x_4, y_5, y_6\}, \{y_1, x_2, x_3, y_4, y_5, y_6\}.$$

Notice that γ_i contains d facets. To prove the next proposition, we recall a theorem from Danaraj and Klee [2]:

Theorem 3.8. *If a $(d-1)$-dimensional simplicial complex Δ is shellable and each $(d-2)$-dimensional face of Δ is contained in at most two facets, then Δ is either a combinatorial ball or a combinatorial sphere.*

Proposition 3.9. *The complex $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ is a shellable $(d-1)$-ball.*

Proof: The shelling order for $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ is as follows:

$$\tau_0^1, \tau_1^1, \tau_1^2, \ldots, \tau_1^d, \tau_2^1, \ldots, \tau_j^{j+1}, \tau_{j+1}^{j+1}, \tau_j^{j+2}, \tau_{j+1}^{j+2}, \ldots, \tau_j^d, \tau_{j+1}^d.$$

Note that this is similar to the shelling order of $\cup_{k=0}^{i}\Gamma_k$. Thus we only need to show that the terms after τ_j^{j+1} comply with the rest of the shelling. For $j+1 \leq k \leq d$, we see that the restriction face of τ_{j+1}^k is $r(\tau_{j+1}^k) = \{y_{j+1}, y_{j+k}\}$, and $r(\tau_j^k) = \{y_j, y_{j+k-1}\}$. Because j is defined by the value of d, and because the starting index is determined by the value of k, we see that the restriction face of a facet is completely determined by the facet, and is thus unique. Therefore, the above ordering of facets is indeed a shelling order.

Now consider any $(d-2)$-dimensional face ρ of $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$. Necessarily, ρ contains $(d-1)$-vertices. the final vertex is from the pair of vertices corresponding to the index number that does not appear in ρ. Thus there can be at most two facets of $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ which contain ρ. Therefore, by 3.8, we see that either $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ is a combinatorial ball or a combinatorial sphere. Because $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ is a pure, full-dimensional subcomplex of the boundary complex of a simplicial d-dimensional polytope, we conclude that $\cup_{k=0}^{j}\Gamma_k \cup \gamma_j$ must indeed be a combinatorial $(d-1)$-ball. □

Proposition 3.10. *The complex* $\Gamma_j \cup \gamma_j$ *has the same homology as* $\mathbb{S}^1$*.*

Proof: The complex $\Gamma_j \cup \gamma_j$ consists of $2d$ facets, $\tau_j^1, \tau_j^2, \ldots, \tau_j^d, \tau_{j-1}^2, \tau_{j-1}^3, \ldots, \tau_{j-1}^{j+1}, \tau_{j+1}^{j+1}, \ldots, \tau_{j+1}^d$. Each facet τ_{j-1}^k has exactly two adjacent facets, τ_j^{k-1} and τ_j^{k+1}. Similarly, the facets τ_{j+1}^k have exactly two adjacent facets, τ_j^{k-1} and τ_j^{k+1}. For $1 < k < j+1$, the facets τ_j^k have two adjacent facets, τ_{j-1}^k and τ_{j-1}^{k+1}. For $j+1 < k \leq d$, the facets τ_j^k also have two adjacent facets, τ_{j+1}^{k-1} and τ_{j+1}^k. Finally, the facet τ_j^1 has τ_{j+1}^d and τ_{j-1}^2 as adjacent facets, and the facet τ_j^{j+1} has τ_{j-1}^{j+1} and τ_{j+1}^{j+1} as adjacent facets. Thus the facet ridge graph of $\Gamma_j \cup \gamma_j$ is a $2d$-cycle. Therefore $\Gamma_j \cup \gamma_j$ has the same homology as $\mathbb{S}^1$. □

The next two lemmas demonstrate that $\Gamma_{j-1} \cup \Gamma_j$ and $\Gamma_j \cup \gamma_j$ are both combinatorial manifolds.

Notation 3.11. For a facet $\tau = \{v_1, ..., v_d\}$ of ∂C_d^*, we may encode the vertices as a vector. define v_j to be 1 if v_j has label x, and -1 otherwise. thus the facet $\{x_1, x_2, y_3, x_4, y_5\}$ of ∂C_5^* may be interpreted as

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

From now on, we identify a facet τ of ∂C_d^* by its corresponding vector.

Lemma 3.12. *For odd d, there exists a bijection* $f : \mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$*.*

Proof: For odd d, define the matrix M_d to have entries $m_{k,2k-1} = -1$ for $1 \leq k \leq j$, $m_{k,2k-d-1} = 1$ for $j+1 \leq k \leq d$, and otherwise $m_{a,b} = 0$. For example, when $d = 5$, we see that

$$M_5 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

. For a facet τ of $\mathcal{B}(1,d)$, define the map $f : \mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$ by $f : \tau \mapsto M_d\tau$, where the operation is matrix multiplication on the facets identified as above.

First, we show that M_d maps identified facets of $\mathcal{B}(1,d)$ to identified facets of $\Gamma_{j-1} \cup \Gamma_j$. Facets of $\mathcal{B}(1,d)$ have at most 1 switch, and because $\mathcal{B}(1,d)$ is centrally symmetric, we only need to look at half of the facets. Thus we may assume that $\tau \in \mathcal{B}(1,d)$ has $v_1 = x_1$. If τ has no switches, then it is clear that $f(\tau) = \{y_1, y_2, y_3, \ldots, y_j, x_{j+1}, x_{j+2}, \ldots, x_d\}$, which is in Γ_j. On the other hand, if τ has a switch, let $k \geq 1$ be the index of said switch. Then it is easy to see that for odd k, $\kappa(f(\tau)) = j$, and for even k, $\kappa(f(\tau)) = j-1$. Also, if $k \neq d-1$, then $f(\tau)$ has exactly two switches, and when $k = d-1$, $f(\tau)$ has exactly one switch, at index $j-1$. Thus $f(\tau)$ has at most two switches, and has $\kappa(f(\tau)) = j-1$ or j. Therefore f maps $\mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$.

Next, we show that M_d has nonzero determinant. because M_d has a gentle slope of -1 entries on top, we can expand by minors to see that

$$\det(M_d) = (-1)^j \det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right) = (-1)^j \det(I) = (-1)^j \neq 0.$$

Since $|\mathcal{B}(1,d)| = |\Gamma_{j-1} \cup \Gamma_j| = 2d$, $f : \mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$ must be a bijection. □

Lemma 3.13. *For even d, there exists a bijection $f : B(1,d) \to \Gamma_j \cup \gamma_j$.*

Proof: Define M_d to be a matrix with entries given by the following: $m_{k,2k-1} = -1$ for $1 \leq k \leq j$, $m_{k,2k-d} = 1$ for $j+1 \leq k \leq d$, and otherwise $m_{a,b} = 0$. For example, when $d = 6$,

$$M_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a facet τ of $\mathcal{B}(1,d)$, define the map $f : \mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$ by $f : \tau \mapsto M_d\tau$, where the operation is matrix multiplication on the facets identified as above.

First, we show that M_d maps identified facets of $\mathcal{B}(1,d)$ to identified facets of $\Gamma_{j-1} \cup \Gamma_j$. Facets of $\mathcal{B}(1,d)$ have at most 1 switch, and because $\mathcal{B}(1,d)$ is centrally symmetric, we begin by looking at half of the facets. Assume that $\tau \in \mathcal{B}(1,d)$ has $v_1 = x_1$. If τ has no switches, then it is clear that $f(\tau) = \{y_1, y_2, y_3, \ldots, y_j, x_{j+1}, x_{j+2}, \ldots, x_d\}$, which is in Γ_j. On the other hand, if τ has a switch, let $k \geq 1$ be the index of said switch. Then it is easy to see that for odd k, $\kappa(f(\tau)) = j+1$, and for even k, $\kappa(f(\tau)) = j$. Also, $f(\tau)$ has exactly two switches. Thus $f(\tau)$ has at most two switches, and has $\kappa(f(\tau)) = j-1$ or j.

For the d facets that we did not consider, let $-\tau$ denote the facet antipodal to τ, and note that $f(-\tau) = -f(\tau)$. Thus $f(\tau)$ has at most two switches, and has $\kappa(f(\tau)) = j$, $j-1$, or $j+1$. Therefore f maps $\mathcal{B}(1,d) \to \Gamma_{j-1} \cup \Gamma_j$.

Expanding M_d by minors along the first j rows yields $\det(M_d) = (-1)^j \det(I) = (-1)^j \neq 0$. Since $|\mathcal{B}(1,d)| = |\Gamma_j \cup \gamma_j| = 2d$, we conclude that $f : \mathcal{B}(1,d) \to \Gamma_j \cup \gamma_j$ must be a bijection.
□

Remark 3.14. both types of maps described above map facets of ∂C_d^* to facets of ∂C_d^*. Because these bijections preserve facets, we see that $\Gamma_{i-1} \cup \Gamma_i$ and $\Gamma_i \cup \gamma_i$ are combinatorial $(d-1)$-manifolds.

3.2 Construction

We begin this section by defining the combinatorial balls that we will use in the triangulation.

Definition 3.15. For odd d, define

$$D_1 := (\cup_{k=0}^{j}\Gamma_k) * \{x_{d+1}\} \text{ and } D_2 := (\cup_{k=0}^{j-2})^c * \{y_{d+1}\}.$$

For even d, define

$$D_1 := (\cup_{k=0}^{j}\Gamma_k \cup \gamma_j) * \{x_{d+1}\} \text{ and } D_2 := ((\cup_{k=0}^{j-1}\Gamma_k)^c \cup \gamma_j) * \{y_{d+1}\}.$$

' Regardless of the parity of d, by 3.5 and 3.9, we know that D_1 and D_2 are both combinatorial d-balls.

Remark 3.16. When d is odd, $D_1 \cap D_2 = \Gamma_{j-1} \cup \Gamma_j$. When d is even, $D_1 \cap D_2 = \Gamma_j \cup \gamma_j$. In either case, by 3.6, 3.10, and 3.14, $D_1 \cap D_2 \subseteq \partial D_1 \cap \partial D_2$ is a combinatorial $(d-1)$-manifold that has the same homology as $\mathbb{S}^1$. This also means that $D_1 \cap D_2$ is path connected.

Since $D_1 \cup D_2$ is a submanifold of ∂C^*_{d+1} on the vertices $x_1, \ldots, x_{d+1}, y_1, \ldots, y_{d+1}$, we see that $\partial(D_1 \cup D_2)$ is a $(d-1)$-dimensional submanifold of ∂C^*_{d+1}, which is a d-sphere.

The above discussion shows that D_1 and D_2 satisfy the hypotheses of 3.2. Therefore, we conclude that $\partial(D_1 \cup D_2)$ is a $(2d+2)$-vertex triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for all $d \geq 5$.

4 Generalizing the Construction

We will close by using induction to generate triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-3}$, where $1 < i \leq \frac{d-1}{2}$. To do this, we need to define a sequence of complexes.

Definition 4.1. Define $D_1^2 := D_1$ and $D_2^2 := D_2$ as in section 3.2, and define

$$D_1^i := (\mathrm{st}(y_{d+i-2}) \cup D_1^{i-1}) * \{x_{d+i-1}\} \text{ and } D_2^i := (\mathrm{st}(x_{d+i-2}) \cup D_2^{i-1}) * \{y_{d+i-1}\}.$$

Proposition 4.2. *There exists a $(2d+2i-2)$-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$, where $1 < i \leq \frac{d-1}{2}$.*

Proof: We induct on i. For $i = 2$, we showed that the base case holds for all $d \geq 5$. Fix $d \geq 2i+3$, and suppose for the sake of induction that in ∂C^*_{d+i-2}, D_1^i and D_2^i are two combinatorial $(d+i-2)$-balls which satisfy the hypotheses of 3.2. because $\{x_{d+i-1}\}$ and $\{y_{d+i-1}\}$ are a pair of antipodal vertices, we may regard D_1^i, D_2^i as submanifolds of ∂C^*_{d+i-1}. Thus defining D_1^{i+1} and D_2^{i+1} makes sense.

Since D_1^i, $\mathrm{st}(x_{d+i-1})$, and their intersection $\cup_{k=0}^{j}\Gamma_k$ are all combinatorial $(d+i-2)$ balls, so is $\mathrm{st}(y_{d+i-2}) \cup D_1^i$. Thus D_1^{i+1} is a combinatorial $(d+i-1)$-ball. Similarly, D_2^{i+1} is a combinatorial $(d+i-1)$-ball. Note that $D_1^{i+1} \cap D_2^{i+1} = D_1^i \cup D_2^i \subseteq (\partial D_1^{i+1} \cap \partial D_2^{i+1})$, which by the inductive hypothesis, is a combinatorial $(d+i-2)$-manifold which has the same homology as S^i. Thus $D_1^{i+1} \cup D_2^{i+1}$ is a combinatorial $(d+i-1)$-manifold, which means $\partial(D_1^{i+1} \cup D_2^{i+1}$ is a $(d+i-2)$-submanifold of ∂C^*_{d+i-1}. Therefore, by 3.2, we see that $\partial(D_1^{i+1} \cup D_2^{i+1})$ triangulates $\mathbb{S}^{i+1} \times \mathbb{S}^{d-i-2}$, completing the proof. □

A future paper in this topic might try to examine some properties of this construction, such as the face values and automorphism group of these complexes. This would give a way to compare Klee and Novik's construction to this one.

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