GAUGED LINEAR SIGMA MODEL SPACES

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Abstract. The gauged linear sigma model (GLSM) originated in physics but it has recently made it into mathematics as an enumerative theory of critical loci. We will study the geometry of the input data of the GLSM, here referred to as GLSM space. We will show that GLSM spaces with a Riemann surface define a toric variety, which allows us to classify all GLSM spaces with a Riemann surface. We will also investigate other examples of GLSM spaces, some of which involve vector bundles.

1. Introduction

The gauged linear sigma model (GLSM) was introduced into physics by Witten in [5] as a special type of a quantum field theory. One of the applications of the GLSM is the Landau-Ginzburg/Calabi-Yau correspondence in physics. By varying the parameters of the GLSM, Witten argued that the GLSM converges to a nonlinear sigma model at a certain limit of the parameters and a Landau-Ginzburg orbifold at a different limit. In mathematics, the GLSM can be viewed as an enumerative theory of critical loci (see [2] for the mathematical theory of the GLSM). In this paper we will study the geometry of the input data of the GLSM, here referred to as GLSM space.

One of the ingredients of a GLSM space is a complex manifold under a holomorphic action of the multiplicative group $\mathbb{C}^*$, although for usual enumerative geometry, GLSM spaces would include a smooth algebraic variety instead. In Section 2, we will classify all GLSM spaces where the complex manifold is a Riemann surface. To do this, we will study the orbits of the $\mathbb{C}^*$-action and the number of fixed points under such action, leading to the observation that non-trivial GLSM spaces with a Riemann surface must be toric varieties. In Section 3, we will begin by giving a criterion to create new GLSM spaces from existing ones. We will then explore other examples of GLSM spaces where X is a complex manifold of dimension greater than one. These examples involve vector bundles, so some background in vector bundles is provided. We will

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also introduce the concepts of degeneracy locus of a GLSM space and of isomorphisms between GLSM spaces, and we will end with an open question about whether two types of GLSM spaces are isomorphic.

Throughout this paper, dimension will always mean complex dimension.

Definition 1. Let X be a complex manifold. A *holomorphic* $\mathbb{C}^*$*-action* on X is a holomorphic function $\mu : \mathbb{C}^* \times X \to X$ satisfying the group action axioms, that is

$$\mu(1, x) = x \qquad \text{and} \qquad \mu(\lambda_1\lambda_2, x) = \mu(\lambda_1, \mu(\lambda_2, x))$$

for all $x \in X$ and $\lambda_1, \lambda_2 \in \mathbb{C}^*$.

In other words, the action μ is a holomorphic group action of the Lie group $\mathbb{C}^*$, the multiplicative group of complex numbers, on a complex manifold X.

Example 1 (Identity $\mathbb{C}^*$-action)**.** Let X be a complex manifold. The holomorphic $\mathbb{C}^*$-action given by $\mu(\lambda, x) = x$ for all $\lambda \in \mathbb{C}^*$ and $x \in X$ will be called the identity $\mathbb{C}^*$-action. Under this action, all points of X are fixed points.

Example 2 (Natural $\mathbb{C}^*$-action)**.** An example of a holomorphic $\mathbb{C}^*$-action on $\mathbb{C}$ is given by $\mu(\lambda, x) = \lambda x$ for all $\lambda \in \mathbb{C}^*$ and $x \in \mathbb{C}$. This will be called the natural $\mathbb{C}^*$-action on $\mathbb{C}$.

Example 3. Another example of a holomorphic $\mathbb{C}^*$-action on $\mathbb{C}$ is given by $\mu(\lambda, x) = \frac{x}{\lambda}$ for all $\lambda \in \mathbb{C}^*$ and $x \in \mathbb{C}$.

Definition 2. A *GLSM space*[1] (X, W, μ) of weight k is a complex manifold X together with a holomorphic function (also called *superpotential*) $W \colon X \to \mathbb{C}$, and a holomorphic $\mathbb{C}^*$-action

$$\begin{aligned} \mu\colon \mathbb{C}^* \times X &\to X \\ (\lambda, x) &\mapsto \lambda.x \end{aligned}$$

such that W has $\mathbb{C}^*$-weight one, that is

$$W(\mu(\lambda, x)) = \lambda W(x)$$

for all $\lambda \in \mathbb{C}^*$ and $x \in X$.

Definition 3. A *GLSM space of weight* k is a GLSM space (X, W, μ) except that W has $\mathbb{C}^*$-weight k, that is

$$W(\mu(\lambda, x)) = \lambda^k W(x)$$

for all $\lambda \in \mathbb{C}^*$ and $x \in X$.

[1] called input data of the GLSM in [2]

In this paper, a GLSM space will mean a GLSM space of weight one, unless specified.

Definition 4. An *algebraic GLSM space* is a GLSM space (X, W, μ) such that X is a smooth algebraic variety, W is algebraic, and μ is an algebraic $\mathbb{C}^*$-action.

Remark 1. We can also generalize the definition of GLSM spaces to allow X to be an orbifold.

Remark 2. In most of the literature, the variety X takes the form of a GIT quotient $V//G$ of a vector space V by an algebraic group G. For example, X can be a proper (compact) toric variety. In [2, Section 3.2.2], the input data for the GLSM is formulated in terms of a vector space V, both a G- and a $\mathbb{C}^*$-action on V, and a G-invariant holomorphic function $W\colon V \to \mathbb{C}$, satisfying several conditions. By G-invariance, W descends to a holomorphic function on the quotient $V//G$. By one of the conditions, the $\mathbb{C}^*$-action on V can be used to define a $\mathbb{C}^*$-action on $V//G$.

Example 4 (Trivial GLSM space)**.** For any complex manifold X and any action μ, we may form the trivial GLSM space $(X, 0, \mu)$, where the superpotential is identically zero.

Example 5. We may also form a GLSM space of weight one where $X = \mathbb{C}$, the $\mathbb{C}^*$-action is the natural action, and the superpotential is the function defined by a degree one homogeneous polynomial over $\mathbb{C}$. That is,

$$\mu(\lambda, x) = \lambda x \;\text{ and }\; W(x) = ax \text{ for some } a \in \mathbb{C}.$$

Example 6. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function. Define the superpotential and the $\mathbb{C}^*$-action by

$$\begin{aligned} W : \mathbb{C}^2 \times \mathbb{C} &\to \mathbb{C} & \qquad \mu : \mathbb{C}^* \times (\mathbb{C}^2 \times \mathbb{C}) &\to \mathbb{C}^2 \times \mathbb{C} \\ (\vec{x}, p) &\mapsto pf(\vec{x}) & (\lambda, (\vec{x}, p)) &\mapsto (\vec{x}, \lambda p). \end{aligned}$$

Then $(\mathbb{C}^2 \times \mathbb{C}, W, \mu)$ is another example of a GLSM space of weight one.

2. GLSM spaces with a Riemann surface

In this section we will classify all GLSM spaces (X, W, μ) such that X is a one-dimensional complex manifold, that is, X is a Riemann surface.

Lemma 1. *Let μ be a non-trivial holomorphic $\mathbb{C}^*$-action on a Riemann surface X. Then the orbits of non-fixed points are isomorphic to $\mathbb{C}^*$ as complex manifolds.*

Proof. Let $x \in X$ be a point which is not fixed under μ. We will use $\mathrm{Stab}_\mu(x)$ and $\mathrm{Orb}_\mu(x)$ to denote the stabilizer group and the orbit of x under μ, respectively. The only possible stabilizer subgroups of x are $\mathbb{C}^*$ or the nth roots of unity, denoted by ω_n. If $\mathrm{Stab}_\mu(x) = \mathbb{C}^*$, then x would be a fixed point, so it must be the case that $\mathrm{Stab}_\mu(x) = \omega_n$ for some $n \in \mathbb{N}$. Define a new $\mathbb{C}^*$-action $\widetilde{\mu}$ on X by $\widetilde{\mu} := \mu(\lambda^n, x)$. This new action $\widetilde{\mu}$ is holomorphic.

Next, we show that $\mathrm{Orb}_\mu(x) = \mathrm{Orb}_{\widetilde{\mu}}(x)$. Let $a \in \mathrm{Orb}_\mu(x)$, so $a = \mu(\lambda_a, x)$ for some $\lambda_a \in \mathbb{C}^*$. By letting $\lambda_\alpha = \lambda_a^{1/n}$, where $\lambda_a^{1/n}$ an nth rooth of λ_a, we see that $a = \widetilde{\mu}(\lambda_a, x)$ and $a \in \mathrm{Orb}_{\widetilde{\mu}}(x)$. Now let $b \in \mathrm{Orb}_{\widetilde{\mu}}(x)$, so $b = \widetilde{\mu}(\lambda_b, x)$ for some $\lambda_b \in \mathbb{C}^*$. By letting $\lambda_\beta = \lambda_b^n$ we see that $b = \mu(\lambda_\beta, x)$ and $b \in \mathrm{Orb}_\mu(x)$.

We also have that $\mathrm{Stab}_{\widetilde{\mu}}(x) = \omega_n^n = \{1\}$, that is, the stabilizer of x under μ is trivial. We can define a function

$$\begin{aligned} \phi : \mathrm{Orb}_{\widetilde{\mu}}(x) &\to \mathbb{C}^* \\ \lambda.x &\mapsto \lambda, \end{aligned}$$

which is a biholomorphism, i.e. a holomorphic bijective function whose inverse ϕ^{-1} is also holomorphic; biholomorphisms are isomorphisms in the category of complex manifolds. Since $\mathrm{Orb}_\mu(x) = \mathrm{Orb}_{\widetilde{\mu}}(x)$, it follows that $\mathrm{Orb}_\mu(x)$ is isomorphic to $\mathbb{C}^*$ as a complex manifold. □

Definition 5. A *toric variety* is an algebraic variety X that contains an algebraic torus T as a dense open subset, together with an action of T on X that extends the natural action of T on itself.

Theorem 1 (Open mapping theorem)**.** *Let X and Y be connected Riemann surfaces and let $f : X \to Y$ be a non-constant holomorphic mapping. Then f is open.*

Proof. See [3, Corollary 2.4]. □

Theorem 2. *Let μ be a non-trivial holomorphic $\mathbb{C}^*$-action on a connected Riemann surface X. Then X is a toric variety.*

Proof. Let $x \in X$ be a non-fixed point under μ. By Lemma 1, $\mathrm{Orb}_\mu(x)$ is isomorphic to $\mathbb{C}^*$ as a complex manifold, so X contains an algebraic torus. Define $U := \mathrm{Orb}_\mu(x)$ and consider the restriction of μ to x, written as $\mu_x : \mathbb{C}^* \to X$. By the open mapping theorem, we get that U is open. We now prove that U is dense in X, i.e. $\overline{U} = U$.

First, we show that $\overline{U} \setminus U$ consists of fixed points. Assume that $v \in \overline{U} \setminus U$ is not a fixed point, i.e. $v = \lambda.y$ for some $\lambda \in \mathbb{C}^*$ and some non-fixed point $y \in X$. Denote by U' the orbit of y. By the open mapping theorem, U' is open. Note that given any two orbits of non-fixed points, then either the orbits are disjoint or they are equal; if the orbits intersect, then they must be equal. Then $U \cap U' = \emptyset$, which contradicts the fact that v is a limit point of U, so v must be a fixed point.

We now show that $\overline{U}$ is open in X, which implies that U is dense in X; otherwise, it would contradict the connectedness of X because $\overline{U}$ would be both open and closed in X. To show that $\overline{U}$ is open, we will find a neighborhood contained in $\overline{U}$ for every point in U. If $u \in U$, then U is a neighborhood of u. If $v \in \overline{U} \setminus U$, then let V be a neighborhood of v homeomorphic to the open unit disk D in $\mathbb{C}$, such that $v \mapsto 0$ under this homeomorphism (the existence of V is guaranteed because X is a complex manifold). Define $W := \mu_x^{-1}(V)$. We now extend μ_x holomorphically to some $p \in \overline{W}$ by setting $\mu_x(p) = v$. It is possible to holomorphically extend μ_x in this way because holomorphic functions "blow up" near singularities; if μ_x had a pole or an essential singularity at p, then the restriction of μ_x to $\overline{W}$ would be unbounded, contradicting the fact that V is homeomorphic to the open unit disk. Note also that it must be the case that $p \in \{0, \infty\}$ because otherwise we would have that $\overline{W} \subset \mathbb{C}^*$ and $\mu_x(p) = \emptyset$. By the open mapping theorem, the restriction of μ_x to $\overline{W}$, written as $\mu_x|_{\overline{W}} : \overline{W} \to \overline{V}$, is an open map. Since $\overline{V} \subset \overline{U}$ and $\overline{V}$ is open, we can conclude that every point in $\overline{U}$ has a neighborhood contained in $\overline{U}$. Therefore, $\overline{U} = X$, i.e. U is dense in X, and X is a toric variety. □

Corollary 1. *Since the only one-dimensional toric varieties are $\mathbb{C}^*$, $\mathbb{C}$, and $\mathbb{CP}^1$, all non-trivial GLSM spaces with a Riemann surface will occur in these three varieties.*

Theorem 3. *Any holomorphic function from a compact Riemann surface to $\mathbb{C}$ is constant.*

Proof. See [3, Corollary 2.8]. □

Corollary 2. *Let $(\mathbb{CP}^1, W, \mu)$ be a GLSM space. Since $\mathbb{CP}^1$ is a compact Riemann surface, the superpotential W will be constant. In addition, since any GLSM space must satisfy that $W(\mu(\lambda, x)) = \lambda W(x)$ for all $\lambda \in \mathbb{C}^*$ and $x \in X$, it follows that the superpotential must be identically zero. Therefore, all GLSM spaces with $\mathbb{CP}^1$ will be trivial.*

Theorem 4 ([1]). *Suppose a multiplicative group G acts on a complex algebraic variety X, and let X^G denote the set of fixed points of X*

under the action of G. We then have the following equality between Euler characteristics: $\chi(X) = \chi(X^G)$.

Lemma 2. *Any non-trivial holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^*$, $\mathbb{C}$, and $\mathbb{CP}^1$ will have exactly 0, 1, and 2 fixed points, respectively.*

Proof. We know that the Euler characteristics of $\mathbb{C}^*$, $\mathbb{C}$, and $\mathbb{CP}^1$ are 0, 1, and 2, respectively. By Theorem 4 we obtain that

$$\chi({\mathbb{C}^*}^{\mathbb{C}^*}) = 0, \qquad \chi(\mathbb{C}^{\mathbb{C}^*}) = 1, \qquad \text{and} \qquad \chi({\mathbb{CP}^1}^{\mathbb{C}^*}) = 2.$$

In the proof of Theorem 2 we showed that the preimage of any fixed point under the holomorphic extension of μ_x can only be 0 or ∞. Hence, the number of fixed points in a Riemann surface under a non-trivial holomorphic $\mathbb{C}^*$-action must be at most 2. We also know that the Euler characteristic of a singleton set is 1, and that the Euler characteristic of a two-point set is 2 (note also that the Euler characteristic is a homotopy invariant). It then follows that

$$\left|{\mathbb{C}^*}^{\mathbb{C}^*}\right| = 0, \qquad \left|\mathbb{C}^{\mathbb{C}^*}\right| = 1, \qquad \text{and} \qquad \left|{\mathbb{CP}^1}^{\mathbb{C}^*}\right| = 2.$$

□

Theorem 5. *The table below classifies all non-trivial GLSM spaces (X, W, μ) where X is a connected Riemann surface.*

	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}$
Number of fixed points	0	0	1
$\mu(\lambda, x) =$	λx	$\frac{x}{\lambda}$	$\lambda(x-b)+b$
$W(x) =$	ax	$\frac{a}{x}$	$a(x-b)$
	$\lambda, a \in \mathbb{C}^*$ *and* $b \in \mathbb{C}$		

Proof. By Corollaries 1 and 2 we know that all such non-trivial GLSM spaces will occur on $\mathbb{C}^*$ and $\mathbb{C}$. By Lemma 2 we know the number of fixed points on each of these spaces.

Let X be $\mathbb{C}^*$ or $\mathbb{C}$, and let μ be the natural action. We can then produce all remaining holomorphic $\mathbb{C}^*$-actions $\widetilde{\mu}$ on X by using the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^* \times X \arrow[r, "\mu"] \arrow[d, "(\mathbb{1},\, \varphi)"'] & X \arrow[d, "\varphi"] \\
\mathbb{C}^* \times X \arrow[r, "\widetilde{\mu}"'] & X
\end{tikzcd}$$

where φ denotes an automorphism of X and $\mathbb{1}$ is the identity.

Let $X = \mathbb{C}$. The automorphisms of $\mathbb{C}$ are of the form $\varphi(x) = ax + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$. Thus, $\varphi^{-1}(x) = \frac{x-b}{a}$. Using the commutative diagram, we obtain that for some $\lambda \in \mathbb{C}^*$ and some $x \in \mathbb{C}$,

$$\varphi \circ \mu \circ (\mathbb{1}, \varphi^{-1})(x) = \lambda(x - b) + b.$$

Therefore, $\widetilde{\mu}(\lambda, x) = \lambda(x - b) + b$ is the general form of a non-trivial holomorphic $\mathbb{C}^*$-action on $\mathbb{C}$, where $b \in \mathbb{C}$. By using the condition that $W(\mu(\lambda, x)) = \lambda W(x)$ for all $\lambda \in \mathbb{C}^*$ and $x \in X$, it follows that

$$W(x) = a(x - b)$$

is the general form of the superpotential W, where $a \in \mathbb{C}^*$.

Let $X = \mathbb{C}^*$. The automorphisms of $\mathbb{C}^*$ are either $x \mapsto ax$ or $x \mapsto \frac{a}{x}$, where $a \in \mathbb{C}^*$. If $\varphi(x) = ax$ for some $a \in \mathbb{C}^*$, so $\varphi^{-1}(x) = \frac{x}{a}$, then the commutative diagram tells us that

$$\widetilde{\mu}(x) = \varphi \circ \mu \circ (\mathbb{1}, \varphi^{-1})(x) = \lambda x.$$

By the condition that $W(\mu(\lambda, x)) = \lambda W(x)$, we obtain that

$$W(x) = ax,$$

where $a \in \mathbb{C}^*$. If $\varphi(x) = \frac{a}{x}$ for some $a \in \mathbb{C}^*$, so $\varphi^{-1}(x) = \varphi(x)$, then we get from the commutative diagram that

$$\widetilde{\mu}(x) = \varphi \circ \mu \circ (\mathbb{1}, \varphi^{-1})(x) = \frac{x}{\lambda}.$$

And by the condition that $W(\mu(\lambda, x)) = \lambda W(x)$, it follows that

$$W(x) = \frac{a}{x},$$

where $a \in \mathbb{C}^*$. □

Remark 3. In the table above, b would be the fixed point on $\mathbb{C}$.

3. More examples of GLSM spaces

In this section we will explore examples of GLSM spaces where X is a complex manifold of dimension greater than one. We begin the section by showing how to form a new GLSM space from an existing one.

Proposition 1. *Let (X, W, μ) be a GLSM space of weight k, and let Y be a submanifold of X. Then (Y, W, μ) is a GLSM space of weight k if and only if for all $z \in X \setminus Y$, we have $\mu^{-1}(\mathrm{Orb}_\mu(z)) \subset X \setminus Y$.*

Proof. Let $x \in X$ be a non-fixed point under μ, so there exists another $x_0 \in X$ such that $x = \mu(\lambda, x_0)$ for some $\lambda \in \mathbb{C}^*$. If we remove x from X to form Y, i.e. $x \in X \setminus Y$, but $x_0 \in Y$, then μ would be undefined on Y, since $\mu(\lambda, x_0) = \emptyset$. □

In other words, to form a new GLSM space using a submanifold Y of X, we may only remove preimages of orbits in X. In particular, we can remove fixed points from X to form Y.

Corollary 3. *Let $(\mathbb{C}, W, \mu)$ be a non-trivial GLSM space. By Theorem 5, we have $\mu(\lambda, x) = \lambda(x-b)+b$ for some $b \in \mathbb{C}$. Then $(\mathbb{C} \setminus \{b\}, W, \mu)$ is a GLSM space if and only if $b = 0$.*

Definition 6. A *holomorphic line bundle* is a holomorphic vector bundle of rank one, that is, a holomorphic vector bundle where the fibers are one-dimensional complex vector spaces.

Definition 7. Let $\pi : E \to X$ be a holomorphic vector bundle of rank r, where X is a complex manifold. Let $E^\vee$ denote the dual of E. We define $\pi^\vee : E^\vee \to X$ to be the *dual vector bundle* of E, which is a holomorphic vector bundle of rank r such that the fibers of $E^\vee$ are the dual vector space to the fibers of E, that is

$$(E^\vee)_x = (E_x)^\vee \text{ for all } x \in X.$$

Definition 8. Let $p : E \to Y$ be a holomorphic vector bundle and let $f : X \to Y$ be a holomorphic function between complex manifolds. The *pullback bundle* $\pi : f^*E \to X$ is defined to be a holomorphic vector bundle such that

$$f^*E = \{(x, e) \in X \times E \mid f(x) = p(e)\} \subset X \times E.$$

We endow f^*E with the subspace topology and let π be the projection onto X, i.e. $\pi(x, e) = x$. In addition, any section s of E over Y induces a *pullback section* f^*s of the pullback bundle by letting

$$f^*s := s \circ f,$$

i.e. $f^*s(x) = (x, s(f(x)))$ for all $x \in X$.

Definition 9. Define

$$\mathcal{O}(-1) := \left\{(p, x) \in \mathbb{CP}^k \times \mathbb{C}^{k+1} \mid \exists\, \lambda \in \mathbb{C}^* \text{ such that } \lambda p = x\right\}.$$

Then $\mathcal{O}(-1)$ is a holomorphic line bundle on $\mathbb{CP}^k$ in a natural way (see [4, Proposition 2.2.6]), and it is known as the *tautological line bundle*. Furthermore, we define $\mathcal{O}(1)$ to be the dual of $\mathcal{O}(-1)$, i.e. $\mathcal{O}(1) := \mathcal{O}(-1)^\vee$; we also get that $\mathcal{O}(1)$ is a holomorphic vector bundle on $\mathbb{CP}^k$. Moreover, using the tensor product and the dual we can define more holomorphic line bundles on $\mathbb{CP}^k$ as

$$\mathcal{O}(n) := \begin{cases} \mathcal{O}(1)^{\otimes n} & \text{for } n > 0, \\ \mathcal{O}(-1)^{\otimes n} & \text{for } n < 0. \end{cases}$$

For any $n, m \in \mathbb{Z}$ we get that $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$. By letting $\mathcal{O}(0) = \mathcal{O}$ be the trivial line bundle on $\mathbb{CP}^k$, we obtain that, under the tensor product operation, the set of line bundles of the form $\mathcal{O}(n)$ for $n \in \mathbb{Z}$ forms an abelian group isomorphic to the group $\mathbb{Z}$.

Example 7 (Example 3.2.15 in [2])**.** Consider a smooth quintic threefold given as the zero locus $\{F_5(x) = 0\} \subset Y := \mathbb{CP}^4$ of a homogeneous degree five polynomial F_5 in five variables. We then let $\mathcal{O}(5)$ be the fifth tensor power of the twisting line bundle $\mathcal{O}(1)$. The polynomial F_5 defines a section $s\colon Y \to \mathcal{O}(5)$. We then let $\pi\colon X := \mathcal{O}(-5) \to Y$. Let $(x_0, \dots, x_5, p)$ denote the coordinates on X. Here, $x_i \in \Gamma(\pi^*\mathcal{O}(1))$ and $p \in \Gamma(\pi^*\mathcal{O}(-5))$. We then set $W := p \cdot F_5(x_0, \dots, x_4) \in \Gamma(\mathcal{O}(-5 + 5 \cdot 1)) = \Gamma(\mathcal{O})$, which we may view as a holomorphic function on X. We define μ by scaling p with weight one: $\mu(\lambda, (x_0, \dots, x_4, p)) = (x_0, \dots, x_4, \lambda p)$.

More generally, instead of the quintic threefold, we can consider any hypersurface in projective space.

Example 8. Let Y be a complex manifold, E a holomorphic vector bundle on Y, and $s \in \Gamma(E)$ be a section of E such that the zero locus $Z = Z(s) \subseteq Y$ is smooth and of dimension $\dim(Y) - \mathrm{rk}(E)$. Then define $X := E^\vee$, the dual vector bundle of E. Let $\pi\colon X \to Y$ be the projection map. Then, the pullback section $\pi^*(s)$ is a section of π^*E. Furthermore, there is a tautological section p of $\pi^*E^\vee$. For example, we may identify $\pi^*E^\vee$ with the fiber product $E^\vee \times_Y E^\vee$, and the section $p\colon E^\vee \to E^\vee \times_Y E^\vee$ is defined via $x \mapsto (x, x)$. By pairing the sections $\pi^*s \in \Gamma(\pi^*E)$ and $p \in \Gamma(\pi^*E^\vee)$ of dual vector bundles, we can define the section $W := \langle \pi^*s, p \rangle \in \Gamma(\mathcal{O})$ (here $\mathcal{O}$ denotes the trivial line bundle $X \times \mathbb{C}$), which may may also regard as a holomorphic function on X. We define μ to act on each fiber of $\pi\colon X = E^\vee \to Y$ with weight one, that is, $v \mapsto \lambda \cdot v$. Then (X, W, μ) is a GLSM space of weight one.

In particular, Examples 6 and 7 are special cases of Example 8. In Example 6, we have that $Y = \mathbb{C}^2 \times \mathbb{C}$. In Example 7, we have that $Y = \mathbb{CP}^4$.

Definition 10. The *degeneracy locus* (or critical locus) of a GLSM space (X, W, μ) is the set of zeros of dW.

Remark 4. An additional restriction that could be put on a GLSM space is that the degeneracy locus should be compact.

Example 9. In Example 4, the degeneracy locus is X itself.

Example 10. In Example 6, the function f is a section of the holomorphic vector bundle $\mathbb{C}^2 \times \mathbb{C}$ on $\mathbb{C}^2$. Assume that the zero locus $Z(f)$ is smooth and of dimension equal to one. For instance, let $f(x_1, x_2) = x_1^2 + x_2^2 - 1$, so $Z(f)$ is the unit circle. The degeneracy locus is then the solution to the following system of equations:

$$\begin{aligned} x_1^2 + x_2^2 - 1 &= 0, \\ 2px_1 &= 0, \\ 2px_2 &= 0. \end{aligned}$$

The smoothness assumption on $Z(f)$ means that $\vec{\nabla} f(x_1, x_2) \neq \vec{0}$ whenever $f(x_1, x_2) = 0$. Thus, the solution to the system of equations above is $Z(f)$ itself. Therefore, the degeneracy locus is $Z(f)$ itself, that is, the unit circle.

Example 11. In Example 7, the degeneracy locus is the solution to the following system of equations:

$$\begin{aligned} F_5(x) &= 0, \\ \frac{\partial F_5}{\partial x_i} &= 0, \end{aligned}$$

where $i \in \{0, 1, 2, 3, 4\}$. Again, the condition that the given quintic threefold is smooth means that the zero locus $Z(F_5)$ is smooth. Hence, the degeneracy locus will be $Z(F_5)$ itself, that is, the quintic threefold that we started with.

Example 12. In Example 8, the degeneracy locus is the zero locus $Z \subseteq Y$ viewed as a subset of X via the zero section $Y \to X$.

Definition 11. Let (X, W_1, μ_1) and (Y, W_2, μ_2) be two GLSM spaces. We say that these two GLSM spaces are isomorphic if there exists a biholomorphism $F : X \to Y$ such that

$$W_1(x) = W_2(F(x)) \tag{1}$$

and F is a $\mathbb{C}^*$-equivariant map (with respect to the actions of μ_1 and μ_2), i.e.

$$F(\mu_1(\lambda, x)) = \mu_2(\lambda, F(x)) \tag{2}$$

for all $x \in X$ and $\lambda \in \mathbb{C}^*$.

Remark 5. Equation (1) is equivalent to the statement that

$$W_1(\mu_1(\lambda, x)) = \lambda W_2(F(x)) \tag{3}$$

for all $x \in X$ and $\lambda \in \mathbb{C}^*$. (1) implies (3) because we can substitute (1) into $W_1(\mu_1(\lambda, x)) = \lambda W_1(x)$. (3) implies (1) by letting $\lambda = 1$ in (3).

Definition 12. Let (X, W_1, μ_1) and (Y, W_2, μ_2) be two GLSM spaces. We say that there is a *morphism* between these two GLSM spaces if there exists a holomorphism $F : X \to Y$ such that

$$W_1(x) = W_2(F(x)) \tag{4}$$

and F is a $\mathbb{C}^*$-equivariant map (with respect to the actions of μ_1 and μ_2), i.e.

$$F(\mu_1(\lambda, x)) = \mu_2(\lambda, F(x)) \tag{5}$$

for all $x \in X$ and $\lambda \in \mathbb{C}^*$.

Remark 6. An isomorphism between two GLSM spaces with complex manifolds X and Y is a morphism of GLSM spaces in which the holomorphism $F : X \to Y$ is a biholomorphism.

Conjecture 1. *Let (X, W, μ) be a GLSM space such that W is a Morse–Bott function on X, in the sense that the degeneracy locus is smooth, and that on the degeneracy locus, the Hessian of W has full rank in the normal directions of the degeneracy locus. Assume additionally that the degeneracy locus is compact. Then (X, W, μ) is isomorphic to the GLSM space of Example 8.*

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