Some Ramsey-theoretic statements without the Axiom of Choice

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Abstract

We investigate infinite sets that witness the failure of certain Ramsey-theoretic statements, such as Ramsey's or (appropriately phrased) Hindman's theorem; such sets may exist if one does not assume the Axiom of Choice. We will show very precise information as to where such sets are located within the hierarchy of infinite Dedekind-finite sets.

1 Hindman's theorem without choice

In this section we will examine how an analogous version of Hindman's Theorem might fail without the Axiom of Choice.

Theorem 1.1 (Hindman's Theorem) *If we finitely color $\mathbb{N}$, then there exists a color c and infinite set D, whose elements all have color c, such that every finite sum over D also has color c.*

Hindman's Theorem is equivalent to the following statement:

Theorem 1.2 *If we finitely color $[\mathbb{N}]^{<\aleph_0}$, then there exists a color c and infinite, pairwise disjoint set D whose elements all have color c, such that every finite union over D also has color c.*

Proof: Let $\phi : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ denote the bijection which maps a natural number n to its binary support, that is the indices of the "1s" when the number is written in binary. For instance, 10 which is 1010_2 in binary would be mapped to the set $\{2,4\}$ and 133 which is 10000101_2 in binary would be mapped to the set $\{1,3,8\}$.

First, we will show that the Theorem 1.2 implies Theorem 1.1. Choose a finite coloring $c : \mathbb{N} \to r$ for some $r \in \omega$. This induces a coloring $c' = c \circ \phi^{-1} : [\mathbb{N}]^{<\omega} \to j$. Per Theorem 1.2, we can find an infinite, pairwise disjoint set $S \subset [\mathbb{N}]^{\omega}$ such that $c'[S] = \{d\}$ for some $d \in r$ and, furthermore, for any set x which is a finite union of sets in S, we also have $c'(x) = d$. Let $T = \phi^{-1}[S] \subset \mathbb{N}$. By construction, $c[T] = \{d\}$. Furthermore, note that given any two sets $x, y \in T$, since S is pairwise disjoint, we have that $\phi(x) \cup \phi(y) = \phi(x+y)$. Thus, $c(x+y) = c'(\phi(x) \cup \phi(y)) = d$ because finite unions of sets in S must have color d. Thus, finite sums in T have color d, so the set S satisfies the criteria laid out in Theorem 1.1.

Now, let's show that Theorem 1.1 implies Theorem 1.2. Fix a coloring $c' : [\mathbb{N}]^{<\omega} \to r$ and let $c = c' \circ \phi : \mathbb{N} \to r$ be the induced coloring on $\mathbb{N}$. Per Theorem 1.1, we get a color d and infinite set $T \subset \mathbb{N}$ such that $c[T] = \{d\}$. We shall now inductively construct an increasing sequence s_n such that the set $S = \{\phi(s_n) \mid n \in \omega\}$ satisfies the criteria of

Theorem 1.2. For ease of notation, define $S_n = \{\phi(s_m) \mid m \leq n\}$. Let $s_0 = \min\{T\}$ and so we have $S_0 = \{\phi(s_0)\}$. Given s_n, we construct s_{n+1} as follows. Define $T_n = \{t \in T \mid t > s_n\}$ and $A = \{t \in T_n \mid \phi(t) \cap S_n = \emptyset\}$. If A is non-empty let $s_{n+1} = \min A$. Otherwise, let $t = \min T_n$ and $M = 1 + \max\{\phi(t)\} \in \omega$. Note that for any $m < n$, $\max\phi(s_m) \leq \max\phi(s_n) < M$ and so for any $p \in \mathbb{N}$, if $\min\phi(p) \geq M$, then $\phi(p) \cap S_n = \emptyset$. Next, note that $|\mathscr{P}(M)| = 2^{M-1}$. Since T_n is infinite, by pigeonhole principle we can find some $I \subset M$ such that $E = \{x \in T_n \mid \phi(x) \cap M = I\}$ is infinite. That is we've found an infinite subset of T_n whose elements' binary expansion all agree on the last $M-1$ digits. Let E' be the smallest 2^{M-1} members of E and let $s_{n+1} = \sum E'$.

Since s_{n+1} is a finite sum of elements of T, it has color d and, furthermore, since it is the sum of 2^{M-1} numbers whose binary expansion agree on the last $M-1$ digits, the binary expansion of s_{n+1} is zero on the last $M-1$ digits. Thus, $\min\phi(s_{n+1}) \geq M$, so $\{\phi(s_{n+1})\} \cap S_n = \emptyset$. So, the final set S is an infinite, pairwise disjoint set such that $c'[S] = \{d\}$. Furthermore, since each s_n is either in T or a finite sum of elements of T (and each $t \in T$ is a summand for at most one such s_n), any finite sum from the sequence s_n can be rewritten as a finite sum of elements of T and therefore has color d. Ergo, any finite union of S must also have color d and so the set S satisfies the criterion of Theorem 1.2.

$\square$

Theorem 1.2 generalizes to the following form by choosing a bijection with $\mathbb{N}$:

Theorem 1.3 *If S is an infinite set, and we finitely color $[S]^{<\aleph_0}$, then there is a color c and infinite, pairwise disjoint subset D whose elements all have color c, such that every finite union over D also has color c.*

However, this generalization is dependent on the Axiom of Choice. In this paper, we aim to explore how this generalization fails without the Axiom of Choice. We present the following definitions:

Definition 1.1 *Let X be a set*

1. *X is called finite if there exists an $n \in \omega$ such that X bijects with n.*
2. *X is called A-finite if X cannot be expressed as the disjoint union of two infinite sets.*
3. *X is called B-finite if X has no infinite linearly orderable subsets.*
4. *X is called C-finite if there is no surjection $f : X \to \omega$. Equivalently, X is C-finite if its powerset is D-finite (see below).*
5. *X is called D-finite if there is no injection from ω into X. Equivalently, there is no injection from X into a proper subset of X.*
6. *X is called E-finite if, for no proper subset Y of X there is a surjection $f : Y \to X$.*
7. *X is called H-finite if its finite power set is D-finite.*

Theorem 1.4 *A set X satisfies Theorem 1.3 if and only if it is H-infinite. In particular, Theorem 1.3 is equivalent to the statement that every H-finite set is finite.*

Proof: Suppose a set S is H-infinite, and let $c : [S]^{<\omega} \to r$ be a finite coloring of its finite powerset. Since S is H-infinite, $[S]^{<\omega}$ is D-infinite, so there exists an injection $f : \omega \to [S]^{<\omega}$. Our goal is to construct an injection $g : [\omega]^{<\omega} \to [S]^{<\omega}$ such that $g(a \cup b) = g(a) \cup g(b)$. Once we've found such an injection, we can pull back the coloring c to $c \circ g : [\omega]^{<\omega} \to r$ a coloring of $[\omega]^{<\omega}$. Then, per Theorem 1.2, we can find an infinite set $D \subset \omega$ whose elements all have color d such that every finite union over D also has color d. Since the map g is injective, we have that $g[D]$ is an infinite subset of $[S]^{<\omega}$ whose elements all have color d, and since g preserves finite unions, it also follows that any finite union over $g(D)$ also has color d. Thus the set $g[D]$ satisfies the properties we are looking for.

So, in order to complete this half of the proof, we just need to show that such an injection exists. Define $g_0 : \omega \to [S]^{<\omega}$ by $g(n) = f(n) \setminus \bigcup_{i \in n} f(i)$. By construction, the image of g_0 is pairwise disjoint. Let $I = \{x \in \omega \mid g_0(x) \neq \emptyset\}$. I claim that I is infinite. Suppose for the sake of contradiction that I is finite. Then, $m = \max(I)$ must exist. Let $F = \bigcup_{i \in m} f(u)$. We now have that for every $n > m$, $f(n) \in \mathscr{P}(F)$. But, note that F is the finite union of finite sets, so F is finite and, in particular, $\mathscr{P}(F)$ is finite. Since there are infinitely many natural numbers larger than m, by pigeonhole principle f cannot be injective. This is a contradiction, so we conclude that I must be infinite. Define $g_1 : \omega \to [S]^{<\omega}$ to be the composition of g_0 with a bijection from ω to I. Thus, for all $n \in \omega$, $g_1(n) \neq \emptyset$. Finally, we can define $g : [\omega]^{<\omega} \to [S]^{<\omega}$ by $a \mapsto \cup g_1[a]$. Since g_1 is injective and its image is pairwise disjoint, it follows that g will also be injective and will preserve finite unions. Therefore, we conclude that if a set is H-infinite, it satisfies Theorem 1.3.

Now suppose the set S satisfies Theorem 1.3. Define $c : [S]^{<\omega} \to 2$ by:

$$x \mapsto \lfloor \log_2 |x| \rfloor \bmod 2$$

Per Theorem 1.3, we now get a set $T \subset S$ of color d which is pairwise disjoint and any finite union will also have color d. Suppose there are two sets $x, y \in T$ such that $|x| = |y|$. Since these two sets are disjoint $|x \cup y| = 2|x|$. In particular, this means that $\log_2 |x \cup y| = 1 + \log_2 |x|$, so $|x|$ and $|x \cup y|$ will have different colors. This is a contradiction, so we conclude that the map $f : T \to \mathbb{N}$ defined by $x \mapsto |x|$ is injective. Since the $\mathbb{N}$ bijects with any infinite subset of itself, we can construct a bijection $g : T \to \mathbb{N}$. Thus, $g^{-1} : \mathbb{N} \to [S]^{<\omega}$ is an injection, and so $[S]^{<\omega}$ is D-infinite and S is H-infinite.

□

2 Finiteness Classes

Definition 2.1 *A class $\mathscr{F}$ is said to be a Finiteness Class provided that:*

1. *If a set S is finite, then $S \in \mathscr{F}$.*
2. *If $S \in \mathscr{F}$ and there exists an injection $f : T \to S$, then $T \in \mathscr{F}$.*
3. *$\omega \notin \mathscr{F}$.*

An immediate consequence of this definition is that the class of finite sets is the smallest finiteness class and D-finite is the largest finiteness class, with respect to inclusion. Previous literature establishes that A through E-finite describe finiteness classes.

Lemma 2.1 *H-finite is a finiteness class*

Proof: If a set is finite, its powerset is finite and therefore D-finite, so all finite sets are H-finite. Similarly, since Hindman's Theorem can be proved for ω without choice, it follows that ω is H-infinite. Finally suppose S is H-finite and $f : T \to S$ is an injection. Well, we immediately get an injection $f' : [T]^{<\omega} \to [S]^{<\omega}$ by $x \mapsto f[x]$. Since S is H-finite, $[S]^{<\omega}$ is D-finite. Since D-finite is a finiteness class and $[T]^{<\omega}$ injects into $[S]^{<\omega}$, $[T]^{<\omega}$ must be D-finite and so T is H-finite. Thus, H-finite describes a finiteness class.

$\square$

Previous literature establishes the following relationships between A through E-finite:

$$\begin{tikzcd}
 & & B\text{-}finite \arrow[drr, Rightarrow, bend left] & & \\
Finite \arrow[r, Rightarrow] & A\text{-}finite \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & D\text{-}finite \\
 & & C\text{-}finite \arrow[r, Rightarrow] & E\text{-}finite \arrow[ur, Rightarrow] &
\end{tikzcd}$$

None of these arrows are reversible and finiteness classes without arrows between them (e.g., B-finite and E-finite) are independent. We have found that when H-finite gets added, the diagram is as follows:

$$\begin{tikzcd}
 & & B\text{-}finite \arrow[drr, Rightarrow, bend left] & & \\
Finite \arrow[r, Rightarrow] & A\text{-}finite \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & E\text{-}finite \arrow[r, Rightarrow] & D\text{-}finite \\
 & & C\text{-}finite \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & & H\text{-}finite \arrow[uur, Rightarrow, bend right] &
\end{tikzcd}$$

3 Ramsey's Theorem and Finiteness Classes

In addition to Hindman's Theorem, we can also use Ramsey's Theorem to define a finiteness class.

Theorem 3.1 (Ramsey's Theorem) *If X is a set and $c : [X]^n \to m$ is a finite coloring of the subsets of X of size n, then there exists an infinite $Y \subset X$ such that $[Y]^n$, the subsets of Y of size n, are monochromatic.*

Definition 3.1 *Given X, if for every finite coloring $c : [X]^n \to m$, we can find an infinite set $Y \subset X$ such that $[Y]^n$ is monochromatic, we say that X is R_n-infinite. If a set X is not R_n-infinite, we say it is R_n-finite.*

Lemma 3.1 *For any n, R_n-finite is a finiteness class.*

Proof: First note that if X is a finite set, then it has no infinite subsets so it cannot possibly be $\mathrm{R_n}$-infinite. Thus, every finite set is $\mathrm{R_n}$-finite. Next, note that Ramsey's Theorem holds on ω without choice, so ω is $\mathrm{R_n}$-infinite for all n. Finally, we want to show that if X is $\mathrm{R_n}$-finite and Y injects into X, then Y is $\mathrm{R_n}$-finite. This is equivalent to showing that if X is $\mathrm{R_n}$-infinite and X injects into Y, then Y is $\mathrm{R_n}$-infinite. Well,let $f : X \to Y$ be an injection and $c : [Y]^n \to m$ be a finite coloring. Define $\tilde{f} : [X]^n \to [Y]^n$ by $x \mapsto f[x]$. Since f is an injection, $\tilde{f}$ is an injection and $c \circ \tilde{f}$ is a finite coloring of X. Since X is $\mathrm{R_n}$-infinite, we can find an infinite set $S \subset X$ such that $[S]^n$ is monochromatic with respect to $c \circ tf$. Thus, $T = \tilde{f}(S) \subset Y$ is monochromatic with respect to c. Since f is an injection and S is infinite, it follows that T is infinite. Thus, T is an infinite subset of Y which is monochromatic with respect to c. So, Y is $\mathrm{R_n}$-infinite. Thus, $\mathrm{R_n}$-infinite is a finiteness class.

□

The next natural question is how does the notion of $\mathrm{R_n}$-finite relate to the previously defined finiteness classes. It turns out that $\mathrm{R_n}$-finite is independent from all previously discussed notions of finite. To show this, we will first construct an A-finite, $\mathrm{R_n}$-infinite set. Since all of the previously discussed definitions of finite follow from A-finite, this establishes that none of them can imply $\mathrm{R_n}$-finite. Then, we will construct an H-infinite $\mathrm{R_n}$-finite set, establishing A-, C-, and H-finite are independent from $\mathrm{R_n}$-finite. Next, we will construct an E-infinite $\mathrm{R_n}$-finite set, establishing E-finite and $\mathrm{R_n}$-finite are independent. Finally, we will construct a B-infinite $\mathrm{R_n}$-finite set establishing B-finite and $\mathrm{R_n}$-finite are independent.

We first consider the set of atoms A in the First Fränkel Model—the model generated by a countable set of atoms, the full permutation group, and the ideal of finite sets. In particular, note that this set of atoms is an amorphous set—an A-finite set. However, I claim that this set is $\mathrm{R_n}$-infinite for all n, which establishes that none of the previously discussed notions of finite, which all follow from A-finite, can imply $\mathrm{R_n}$-finite (for all n). To see this, choose a finite coloring $c:[A]^n \to m$. Since this function exists in the model, it must have a finite support $E \subset A$. Let $S = A \setminus E$. I claim that $[S]^n$ is monochromatic, thereby witnessing A as $\mathrm{R_n}$-infinite. To see this, fix $a,b \in [S]^n$ and enumerate them as $a_1,\dots,a_n$ and $b_1,\dots,b_n$. Let π_i denote the transposition which turns a_i into b_i. Since the a_i are distinct, it follows that $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$ is a permutation which will turn a into b. Furthermore, π will not change any atoms not in $a \cup b \subset S$. In particular, π will fix E, so c must be symmetric with respect to π. Since $c(a) \in \omega$ is a pure set, it follows that $\pi(a,c(a)) = (\pi(a),\pi(c(a))) = (b,c(a))$, and so both a and b must be mapped to the same color. Thus, the set $[S]^n$ is monochromatic, and so A is $\mathrm{R_n}$-infinite.

We now consider the set of atoms A in the Second Fränkel Model—the model generated by a countable set of atoms broken into pairs $P_i = \{a_i,b_i\}$, the permutation group generated by transpositions of these pairs, and the ideal of finite sets. Note that ω injects into $[A]^{<\omega}$ by $n \mapsto P_i$, so $[A]^{<\omega}$ is Dedekind-infinite. Thus, A is H-infinite and as a result C-infinite and A-infinite. However, I claim that for all $n>1$, the set A is $\mathrm{R_n}$-finite. Thus, for all $n>1$, $\mathrm{R_n}$-finite is independent from A, C, and H finite. To see this, consider the coloring $c:[A]^n \to 2$ defined by $x \mapsto 0$ if $(\exists y \in P)(|x \cap y| > 1)$ and 1 otherwise, where $P \subset \mathscr{P}(A)$ is the set of pairs. That is, a set is color 0 if it contains two elements from a pair and 1 if it contains at most one element from each pair. Note that c can be defined within the model, so it is a symmetric function. Suppose $Y \subset A$ is an infinite set with support E. Since E is finite, we can find some atom $a_i \in Y$ which is not in E. Since Y is symmetric, it follows that Y must contain the corresponding atom in the pair b_i. As a result, there must be some element in $[Y]^n$ with color 0. However, since each pair is finite and Y is infinite, Y must also contain elements from at least n pairs so $[Y]^n$ must also contain an element of color 1. Thus, no infinite subset of A can be monochromatic, so A is $\mathrm{R_n}$-finite for $n>1$.

Before establishing, for $n>2$, $\mathrm{R_n}$-finite is independent from E-finite, we note that this technique for producing a bad coloring generalizes as follows:

Lemma 3.2 *If X is a set and P is a partition of X (i.e., P is pairwise disjoint and $\bigcup P = X$) such that every element of P is finite and no infinite subset of P admits a choice function, then X is R_n-finite for all $n>2$.*

Proof: For $n>2$, we defined a coloring $c:[X]^n \to 2$ as before:

$$x \mapsto \begin{cases} 0 & (\exists p \in P)(|x \cap p| > 1) \\ 1 & (\forall p \in P)(|x \cap p| \leq 1) \end{cases}$$

Suppose $Y \subset X$ is an infinite subset such that $[Y]^n$ is monochromatic. First note that since each $p \in P$ is finite, the set $Q = \{p \in P \mid p \cap Y \neq \emptyset\}$ is infinite. In particular, Y contains an element from n distinct elements of P and so $[Y]^n$

contains at least one elementof color 1. Furthermore, if P contained two elements from an element of P, then $[Y]^n$ would contain an element of color 0 which would contradict $[Y]^n$ being monochromatic. Thus, for each element $q \in Q$, we have that $|Y \cap q| = 1$. However, this means that the function $f : Q \to \bigcup Q$ defined by $x \mapsto \bigcup(x \cap Y)$ is a choice function, which contradicts P having no infinite subset with a choice function! Thus, there is no infinite $Y \subset X$ with $[Y]^n$ monochromatic, so X is $\mathrm{R_n}$-finite for all $n > 1$.

□

Returning to the Second Fränkel Model, we now define the set $B \subset [A]^{<\omega}$ in order to show E-finite and $\mathrm{R_n}$-finite are independent for all $n > 1$. Define $B_n = \{x \subset \bigcup_{i=0}^{n} P_i \mid (\forall i \in \omega)(i \leq n \Rightarrow |x \cap P_i| = 1)\}$. That is, elements of B_n are sets which contain exactly one element from each pair of index up to n. Define $B = \bigcup_{i \in \omega} B_i$. Define $f : B \setminus B_0 \to B$ by $x \in B_i \mapsto x \cap \bigcup_{i=0}^{n-1} P_i$. Note that f is well defined because the B_i are pairwise disjoint and that f maps B_{i+1} surjectively onto B_i. In particular, f surjects onto B witnessing the set B as E-infinite. Since each pair P_i is finite, it follows thateach B_i is finite and so to show that B is $\mathrm{R_n}$-finite, by Lemma 3.2, we just need to show that there is no choice function on an infinite family of B_i. Well, suppose $\mathscr{B}$ is such a family of B_i and $f : \mathscr{B} \to \bigcup \mathscr{B}$ is a choice function. Let $\mathscr{P} = \{P_i \mid B_i \in B\}$ be the corresponding collection of pairs. We can now define a choice function $g : \mathscr{P} \to \bigcup \mathscr{P}$ by $P_i \mapsto f(B_i) \cap P_i$. However, the Second Fränkel Model is constructed precisel so that the set of atoms is a Russel set and, in particular, no infinite family of pairs admits a choice function. Thus, the choice function f cannot exist, and so by Lemma 3.2 B is $\mathrm{R_n}$-finite for all $n > 1$.

Finally, we want to produce a B-infinite, $\mathrm{R_n}$-finite set.

Theorem 3.2 *An amorphous set is either R_n-finite for all $n > 1$ or R_n-infinite for all $n > 1$.*

Proof: We will show the equivalent statement that, for any $n > 1$, if an amorphous set is $\mathrm{R_n}$-finite, then it is $\mathrm{R_m}$-finite for all other $m > 1$. We will deal first with $\mathrm{R_2}$-finite, which is a special case, and then prove the result for $n \geq 3$ via an inductive proof. Let A be an amorphous set and let $c : [A]^2 \to 2$ be a bad coloring (i.e., a coloring where no infinite subset of A is monochromatic). Per Lemma 3.2, to show that A is $\mathrm{R_m}$-finite for all $m > 1$, we just need to partition A into finite sets such that no infinite collection of these sets admits a choice function. We will now construct such a partition. For each $x \in A$, we define the following sets:

$$F_{0,x} = \{y \in A \setminus \{x\} \mid c(x,y) = 0\}$$
$$F_{1,x} = \{y \in A \setminus \{x\} \mid c(x,y) = 1\}$$
$$F_0 = \{x \in A \mid F_{0,x} \text{ is finite}\}$$
$$F_1 = \{x \in A \mid F_{1,x} \text{ is finite}\}$$

Since, for each x, $A = \{x\} \cup F_{0,x} \cup F_{1,x}$ and $F_{0,x} \cap F_{1,x} = \emptyset$, it must be the case that exactly one of $F_{0,x}$ and $F_{1,x}$ is finite. Thus, $F_0 \cap F_1 = \emptyset$ and $F_0 \cup F_1 = A$. Since A is amorphous, we now conclude that exactly one of F_0 and F_1 are infinite. Without loss of generality, F_0 is infinite.

For each $x \in F_0$, we inductively define the following sets:

$$N_{x,0} = \{x\}$$
$$N_{x,n+1} = \left\{ y \in F_0 \setminus \bigcup_{i=0}^{n} N_{x,i} \;\middle|\; (\exists z \in N_{x,n})\big(c(y,z) = 0\big) \right\}$$
$$N_x = \bigcup_{i=0}^{\infty} N_{x,i}$$

I claim that each N_x must be finite. First, note that each $N_{x,i}$ must be finite: suppose for some x and i, $N_{x,i}$ is finite and $N_{x,i+1}$ is infinite. By definition, for each point $y \in N_{x,i+1}$, there must be some $z \in N_{x,i}$ such that $c(z,y) = 0$. Thus, every point in $N_{x,i+1}$ is contained in $F_{0,z}$ for at least one $z \in N_{x,i}$. Since $N_{x,i}$ is finite, pigeonhole principle tells us that there must be some $w \in N_{x,i}$ with infinite $F_{w,0}$, which contradicts that $N_{x,i} \subset F_0$. Since we always have that $N_{x,0}$ is finite, it follows inductively that every $N_{x,i}$ is finite. Next, note that if, for some j, $N_{x,j} = \emptyset$, then it must be the case that for all $i > j$, that $N_{x,i} = \emptyset$. Finally, note that by construction, for fixed x, the $N_{x,i}$ are all disjoint. Now, suppose that for some $x \in F_0$, N_x is infinite. Since each $N_{x,i}$ is finite, it must be the case that every $N_{x,i}$ is non-empty. In particular, we can now define two infinite disjoint sets $O = \bigcup_{i=0}^{\infty} N_{x,2i+1}$ and $E = \bigcup_{i=0}^{\infty} N_{x,2i}$ such that $F_1 \cup O \cup E = A$. Since A is amorphous, this is a contradiction, so we conclude that every N_x must be finite.

We now define the following equivalence relation on F_0: $a \sim b \Leftrightarrow a \in N_b$.[1] It follows that for each $a \in F_0$, the equivalence class of a, $[a]$, is precisely N_a. Since each N_a is finite while F_0 is infinite, we conclude that $P = \{[a] \mid a \in F_0\}$ is infinite. In particular $P \cup \{F_1\}$ is a partition of A into finite pieces, so, per Lemma 3.2, to show that A is $\mathrm{R_n}$-finite for all $n > 1$, we just need to show that no infinite subset of $P \cup \{F_1\}$ admits a choice function. In particular, it is sufficient to show that no infinite subset of P admits a choice function.

Suppose that $P' \subset P$ is infinite and $f : P' \to \bigcup P'$ is a choice function. Let I be the image of f. I claim that $[I]^2$ is monochromatic of color 1. Suppose that there is some $a, b \in I$ such that $c(a,b) = 0$. Well, since $a, b \in I$, we have that $f(N_a) = a$ and $f(N_b) = b$. In particular, $N_a \neq N_b$ and so $\neg(a \sim b)$. However, if $c(a,b) = 0$, then $a \in N_{b,1} \subset N_b$ so $a \sim b$. This is a contradiction, so $[I]^2$ is monochromatic of color 1. This in turn contradicts that $c : [A]^2 \to 2$ is a bad coloring, so we conclude that no infinite subset of P admits a choice function. Thus, we conclude that A is $\mathrm{R_n}$-finite for all $n > 1$.

We only used the hypothesis that $c : [A]^2 \to 2$ is a bad coloring to show that no infinite subset of P admits a choice function. Another criterion for determining that no infinite subset of P admits a choice function is that $(\forall^* p \in P)(|p| > 1)$. To see this, suppose that P satisfies this condition, but we can still find $P' \subset P$ with choice function $f : P' \to \bigcup P'$. We can now partition A into $I = \{x \in A \mid (\exists p)(x = f(p))\}$ and $J = \{x \in A \mid (\forall p)(x \neq f(p))\}$. Since P' is infinite, I is infinite. However, $\bigcup_{p \in P'} p \setminus \{f(p)\} \subset J$ is also infinite because, by hypothesis, for all but finitely many $p \in P$ (and by extent P'), we have that $|p| > 1$ so $p \setminus \{f(p)\} \neq \emptyset$. This contradicts the fact that A is amorphous, so we conclude that our new criterion is valid.

We call a coloring $c : [A]^2 \to 2$ "dense" if, for all but finitely many $a \in A$, we can find $x, y \in A$ such that $c(a,x) = 0$ and $c(a,y) = 1$. If we construct the partition P with respect to a dense coloring, for all but finitely many points x, we know that $N_{1,x}$ is non-empty, and so all but finitely many of the partitions must be non-singular. Ergo, the partition constructed with respect to a dense coloring will satisfy the criteria for Lemma 3.2. Thus, the existence of a dense coloring also witnesses that A is $\mathrm{R_n}$-finite for all $n > 1$.

Define the following relation on $A \times A$, with respect to a coloring $c : [A]^n \to 2$: we say $x R_{c,k} y$ provided that $\exists z_3, z_4, \ldots, z_n$ such that $c(x, y, z_3, \ldots, z_n) = k$. We say a point x is "universal" with respect to a coloring $c : [A]^n \to 2$ if $(\forall^* y \in A)(\forall k \in 2)(x R_{c,k} y)$. We will now inductively show that for $n \geq 3$, given a coloring $c : [A]^n \to 2$ with a universal point x, we can construct a dense coloring $d : [A]^2 \to 2$. Then, to complete the proof, we will show that given any bad coloring $c : [A]^n \to 2$, we can either find a universal point with respect to the coloring or produce a bad coloring $d : [A]^2 \to 2$.

We now establish the base case for the induction. Suppose u is a universal point with respect to $c : [A]^3 \to 2$. Let $S = \{x \in A \mid (\forall k \in 2)(u R_{c,k} x)\}$ and define the coloring $d : [A]^2 \to 2$ by $d(x,y) = c(u,x,y)$. Fix a point $s \in S$. Since

[1] Verifying this is an equivalence relation is left as an exercise for the reader.

$u\,R_{c,0}\,s$, there exists some $a \in A$ such that $c(u,s,a) = 0$. Likewise, since $u\,R_{c,1}\,s$, there exists some $b \in A$ such that $c(u,s,b) = 1$. Thus, $d(s,a) = 0$ and $d(s,b) = 1$, so for every $s \in S$, we can find $a, b \in A$ such that $d(s,a) = 0$ and $d(s,b) = 1$. Since u is universal, all but finitely many points in A are in S. Thus, d is a dense coloring.

Now, suppose that we know, for some n, that any coloring $d : [A]^n \to 2$ equipped with a universal point induces a dense coloring $e : [A]^2 \to 2$. Let $c : [A]^{n+1} \to 2$ be a coloring equipped with a universal point u. Let $S = \{x \in A \mid (\forall k \in 2)(u\,R_{c,k}\,x)\}$ and define the coloring $d : [A]^n \to 2$ by $d(x_1,\ldots,x_n) = c(u,x_1,\ldots,x_n)$. If a universal point exists with respect to d, then we get a dense coloring $e : [A]^2 \to 2$ by the inductive hypothesis. So, suppose that this coloring does not admit a universal point. For each $x \in A$, we define the following sets:

$$F_{0,x} = \{y \in A \setminus \{x\} \mid x\,R_{d,0}\,y\}$$
$$F_{1,x} = \{y \in A \setminus \{x\} \mid x\,R_{d,1}\,y\}$$
$$F_0 = \{x \in A \mid F_{0,x} \text{ is finite}\}$$
$$F_1 = \{x \in A \mid F_{1,x} \text{ is finite}\}$$

Note that unlike last time, $F_{0,x}$ and $F_{1,x}$ are not necessarily disjoint. However, if both were infinite (and therefore cofinite, since A is amorphous), then their intersection would be infinite, and so x would be universal. Since d does not admit a universal point, for each x, at least one of $F_{0,x}$ and $F_{1,x}$ must be finite. As a result, although F_0 and F_1 may not be disjoint, we know that every point belongs to at least one of them, so $F_0 \cup F_1 = A$ and, in particular, at least one of F_0 and F_1 must be infinite. Without loss of generality, F_0 is infinite.

I now claim that the coloring $e : [A]^2 \to 2$ given by $e(x,y) = 0$ if $x\,R_{d,0}\,y$ and 1 otherwise, is dense. To see this, consider the set $T = S \cap F_0$, which is infinite because both S and F_0 are infinite. Fix a point $t \in T$. Since $t \in F_0$, there exist only finitely many points a such that $e(t,a) = 0$, so there must exist an infinite amount of points b such that $e(t,b) = 1$. However since $t \in S$, $u\,R_{c,0}\,t$, so there exists $z_3,\ldots,z_{n+1}$ such that $c(u,t,z_3,\ldots,z_{n+1}) = 0$. Thus, $d(t,z_3,\ldots,z_{n+1}) = 0$, so $t\,R_{d,0}\,z_3$, so $e(t,z_3) = 0$. Therefore, for every $t \in T$, we can find points b and z_3 such that $e(t,b) = 1$ and $e(t,z_3) = 0$. Since T is a cofinite set, we conclude that the coloring $e : [A]^2 \to 2$ is dense. This completes the induction.

Finally, to complete the proof, we shall show that given a bad coloring $c : [A]^n \to 2$, we can either find a universal point or construct a bad coloring $e : [A]^2 \to 2$. Supposing no universal point exists with respect to the coloring, we repeat the previous construction. For any point $x \in A$, we define the following sets:

$$F_{0,x} = \{y \in A \setminus \{x\} \mid x\,R_{c,0}\,y\}$$
$$F_{1,x} = \{y \in A \setminus \{x\} \mid x\,R_{c,1}\,y\}$$
$$F_0 = \{x \in A \mid F_{0,x} \text{ is finite}\}$$
$$F_1 = \{x \in A \mid F_{1,x} \text{ is finite}\}$$

As before, at least one of F_0 and F_1 must be infinite. Without loss of generality, F_0 is infinite. As before, we define $e : [A]^2 \to 2$ by $e(x,y) = 0$ if $x\,R_{c,0}\,y$ and 1 otherwise. We want to show that this is a bad coloring. Well, suppose we can find an infinite set $I \subset A$ such that $[I]^2$ is monochromatic. We know that for all but finitely many points a, there exist only finitely many other points b such that $a\,R_{c,0}\,b$. That is, for all but finitely many points a, there are only finitely many other points b such that $e(a,b) = 0$. Thus, $[I]^2$ must be monochromatic in color 1. However, I now claim that it must be the case that $[I]^n$ is monochromatic in color 1 with respect to c—contradicting that c is a bad coloring. To see this, suppose that we can find some $x_1,\ldots,x_n \in I$ such that $c(x_1,\ldots,x_n) = 0$. But then, by definition, $x_1\,R_{c,0}\,x_2$ which means $e(x_1,x_2) = 0$ which contradicts that $[I]^2$ is monochromatic of color 1. Thus, $[I]^n$ is monochromatic of color 1 with respect to c, which contradicts that c is a bad coloring. Thus, $e : [A]^2 \to 2$ is a bad coloring. So, we conclude that given any bad coloring $c : [A]^n \to 2$, we can either produce a dense coloring $e : [A]^2 \to 2$ or a bad coloring $e : [A]^2 \to 2$. In both cases, we've shown we can produce a partition satisfying the criterion for Lemma 3.2 which tells us that A is $\mathrm{R_n}$-finite for all $n > 1$. Thus, if A is $\mathrm{R_n}$-finite for any $n > 1$, A is $\mathrm{R_m}$-finite for all $m > 1$. In other words, an amorphous set A is either $\mathrm{R_n}$-finite for all $n > 1$, or $\mathrm{R_n}$-infinite for all $n > 1$. □

4 Hindman's Theorem and Finiteness Classes

Definition 4.1 *A set X is $H_D - Infinite$, if for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite pairwise disjoint $Y \in [X]^{<\omega}$, st, $FU(Y)$ is monochromatic. (Or equivalently, if $[X]^{<\omega}$ is Dedekind Infinite, then X is $H_D - Infinite$.)*

Definition 4.2 *A set X is $H_B - Infinite$, if for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite $Y \in [X]^{<\omega}$, st, $F\triangle(Y)$ is monochromatic.*

Definition 4.3 *A set X is $H_{D,n} - Infinite$, if for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite pairwise disjoint $Y \in [X]^{<\omega}$, st, $FU_{\leqslant n}(Y)$ is monochromatic.*

Definition 4.4 *A set X is $H_{B,n} - Infinite$, if for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite set $Y \in [X]^{<\omega}$, st, $F\triangle_{\leqslant n}(Y)$ is monochromatic.*

Theorem 4.1 *For a set X, the following 4 statements are equivalent:*

① *X is $H_{D,2} - Fin$.*
② *X is $H_D - Fin$.*
③ *$[X]^{<\omega}$ is $D - Fin$.*
④ *X is $H_{B,4} - Fin$.*

$$\begin{tikzcd}
H_D - Infinite \arrow[rr] \arrow[d, "① \to ②"'] & & H_{B,4} - Infinite \arrow[d, "③ \to ④"] \\
H_{D,2} - Infinite \arrow[rr] & & {[X]^{<\omega} \text{ is } D - Infinite} \arrow[ull, "② \to ③"']
\end{tikzcd}$$

To show these 4 statements are equivalent, it is sufficient to show that: ① → ②;② → ③;③ → ④;④ → ①. Each of these 4 implications will be proved by contrapositive.

① → ②:

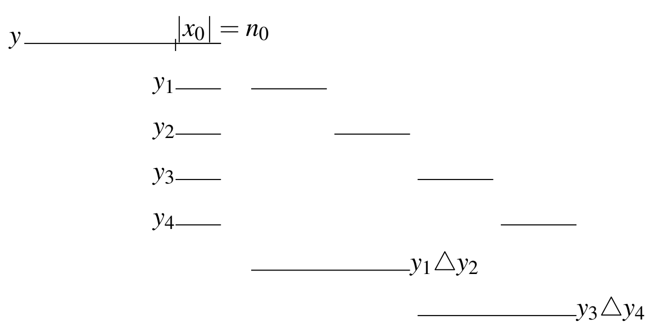
Proof: Suppose X is $H_D - Infinite$, then for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite pairwise disjoint $Y \in [X]^{<\omega}$, st, $FU(Y)$ is monochromatic. Clearly, $FU_{\leqslant 2}(Y)$ is monochromatic, which means X is $H_{D,2} - Infinite$. □

② → ③:

Proof: Suppose $[X]^{<\omega}$ is $D-Infinite$, then there exists an injective map f from $\omega \to [X]^{<\omega}$. Image under this map would be $\{f(1), f(2), \cdots, f(n), \cdots\}$. Make elements of $Im(f)$ pairwise disjoint by following: take $x_1 = f(1)$, since x_1 is a finite set, then it only has finitely many subsets, so there exists a smallest $i \in \omega$, st, $f(i) \setminus f(1) \neq \emptyset$, define $x_2 = f(i) \setminus f(1)$; since $x_1 \cup x_2$ is also finite, then for the same reason above, there exists a smallest $j \in \omega$, st, $f(j) \setminus (x_1 \cup x_2) \neq \emptyset$, define $x_3 = f(j) \setminus (x_1 \cup x_2) \cdots$ continue in this fashion, we will have an infinite set $Y = \{x_1, x_2, \cdots, x_n, \cdots\}$ in bijection with ω, which is pairwise disjoint. Now, take arbitrary coloring map c: $[X]^{<\omega} \to 2$ and restrict it to Y. Since Y is isomorphic to ω, then coloring set Y is equivalent as coloring natural number. Without choice, there is still full Hindman's Theorem on natural number, so there exists infinite set $I \in \omega$, st, $FU(I)$ is monochromatic. Then by isomorphism, there is pairwise disjoint infinite set $Y' = \{x_n | n \in I\}$, st, $FU(Y')$ is monochromatic. Therefore, X is $H_D - Infinite$.

□

③ → ④:



Proof: Suppose X is $H_{B,4} - Infinite$. Then by definition, for all coloring maps c: $[X]^{<\omega} \to 2$, there is an infinite set $Y \in [X]^{<\omega}$, st, $F\triangle_{\leqslant 4}(Y)$ is monochromatic. Here we take a special coloring map c: $[X]^{<\omega} \to 2$ which sending every $x \in [X]^{<\omega}$ to $\lfloor log_2|x| \rfloor$ (mod 2). [*note:* $\forall x \in [X]^{<\omega}$, *write* $|x|$ *in binary form, coloring map c defined above would give us the furthest position of nonzero digit in* $|x|$ *modulo* 2] Then there exists an infinite set $Y \subseteq [X]^{<\omega}$, st, $F\triangle(Y)$ is monochromatic. By lemma, for any cardinality $n \in \omega$, there are only finitely many $y \in Y$, st, $|y| = n$. Construct a set X' with elements $X_1, X_2, ..., X_n, ...$, st, for all i, each set contained in X_i has the same cardinality.X' must be an infinite set, otherwise Y would not be an infinite set. And also, $\bigcup X_i \in [X]^{<\omega}, \forall i$. Now, we have a countable, infinite set $\{\bigcup X_i | i \in \omega\} \subseteq [X]^{<\omega}$, which means $[X]^{<\omega}$ is Dedekind Infinite.

Proof of Lemma (by contradiction) We know that for a fixed cardinality n, there do not exist two sets y, z with cardinality n that satisfy $y \cap z = \emptyset$, otherwise, $y \triangle z$ will not be of the same color as y or z. Thus, any two sets with the same cardinality must have nonempty intersection. (claim)

Suppose there exists $n \in \omega$, st, there are infinitely many sets in Y with cardinality n. Choose arbitrary set $y \in Y$, st, $|y| = n$. Since we have infinitely many sets with the same cardinality, then by our claim above, all the other sets (infinitely many) should intersect y at some place. Since y is finite, then it has only finitely many subsets. By pigeonhole principle, there should be infinitely many sets with cardinality n intersect y at the same place x_0, $|x_0| = n_0$. Now, collect these sets together to be a new set Y'. Define a coloring map c' on $[Y']^2 = \{(a,b) | a, b \in Y'\}$ by the rule that if $(a\setminus x_0)\cap(b\setminus x_0)=\emptyset$, then color the pair red, otherwise, color the pair blue. By Ramsey Theorem, there exist infinitely many pairs in $[Y']^2$ that are monochromatic. And furthermore, this color can not be red. Otherwise, we can take 4 arbitrary sets $y_1,y_2,y_3,y_4\in Y'$, st, every pair contained in $\{(y_i,y_j)|i,j\in\{1,2,3,4\}\}$ is colored red. Then $|y_1\triangle y_2|=|y_3\triangle y_4|=n-n_0$, and $y_1\triangle y_2$,$y_3\triangle y_4$ are disjoint, this contradicts our claim above. Therefore, we have a infinite set $Y''\subseteq Y'$, st, any two sets in Y'' have a bigger intersection than x_0 (in other words, $\forall c,d\in Y'',x_0\subset(c\cap d)$, which implies that $|c\cap d|\geqslant n_0+1$). Continue in this fashion, we can now choose arbitrary $y'\in Y''$ and infinitely many sets in Y'' intersect y' at the same place x_1, $|x_1|\geqslant n_0+1$...

Since the set we start with has finite cardinality n, this procedure will end in at most n steps. (since the cardinality of the intersection parts would increase at least 1 at each step, eventually, Ramsey Theorem would guarantee the existence of infinitely many sets with cardinality n, st, any two of these sets have intersection with cardinality n, which means there is actually only one set. But this result contradicts Ramsey Theorem.) And in each step, instead of requiring the existence of infinitely many red pairs of sets, we only need 4 sets with disjoint "tails" to arrive at a contradiction. Thus we just need to assume there are $2^n+\underbrace{R(4,2^{n-1}+R(4,\cdots R(4,2)\cdots))}_{n-1}$ sets with cardinality n at the beginning of this proof. This is a really big but finite number, so axiom of choice is not involved in this proof.

□

④ $\to$ ①:

Proof: Instead of proving ④ $\to$ ① directly, let's separate it into three pieces: (1) X is $H_{D,2}-Infinite\to[X]^{<\omega}$ is $D-Infinite$; (2) $[X]^{<\omega}$ is $D-Infinite\to X$ is $H_D-Infinite$ (② $\to$ ③); (3) X is $H_D-Infinite\to X$ is $H_{B,4}-Infinite$ (① $\to$ ②). Since (2),(3) have been proved above, we only provide the proof of (1) here. Suppose X is $H_{D,2}-Infinite$, then by definition, for all coloring maps c: $[X]^{<\omega}\to 2$, there is an infinite pairwise disjoint $Y\in[X]^{<\omega}$, st, $FU_{\leqslant 2}(Y)$ is monochromatic. Use the special coloring map c: $[X]^{<\omega}\to 2$ which sending every $x\in[X]^{<\omega}$ to $\lfloor log_2|x|\rfloor$ (mod 2). Then each element in Y should have different cardinality. Otherwise, there would be $|A|=|B|=n$ $(A,B\in Y,n\in\omega)$, since A,B are disjoint, $c(A)=c(B)\neq c(A\cup B)$, which contradicts definition 3. Thus we have an injective map from ω to $[X]^{<\omega}$, which sending $n\in\omega$ to $y\in Y\subset[X]^{<\omega}$, $(|y|=n)$. Therefore, $[X]^{<\omega}$ is $D-Infinite$.

□

Corollary of Theorem 4.1 *$H_B-Infinite$ and $H_D-Infinite$ are equivalent.*

Proof: Suppose X is $H_B-Infinite$, then by definition, for all coloring maps c: $[X]^{<\omega}\to 2$, there is an infinite $Y\in[X]^{<\omega}$, st, $F\triangle(Y)$ is monochromatic. Clearly, $F\triangle_{\leqslant 4}(Y)$ is also monochromatic, which means X is $H_{B,4}-Infinite$ and furthermore $H_D-Infinite$ (by theorem 4.1).

For the other direction, suppose X is $H_D-Infinite$, then by definition, for all coloring maps c: $[X]^{<\omega}\to 2$, there is an infinite pairwise disjoint $Y\in[X]^{<\omega}$, st, $FU(Y)$ is monochromatic. Since Y is pairwise disjoint, then taking symmetric difference is equivalent as taking union, which means $F\triangle(Y)$ is also monochromatic, X is $H_B-Infinite$.

□

Definition 4.5 *A set X is $R_2-Infinite$ if for all coloring maps c: $[X]^{<\omega}\to 2$, there is an infinite $Y\in[X]^{<\omega}$, st, $[Y]^2$ is monochromatic.*

Theorem 4.2 *$H_{B,2}-finite$ implies $R_2-finite$*

Proof: Suppose X is $R_2 - Infinite$. Take arbitrary coloring map c: $[X]^{<\omega} \to 2$, there exists an infinite $Y \in [X]^{<\omega}$, st, $[Y]^2$ is monochromatic. Fix $y_0 \in Y$ and construct $Y' = \{\{y_0, y\} | y \in Y \setminus \{y_0\}\}$. Then $F\triangle(Y') = \{\{y_i, y_j\} | y_i, y_j \in Y \setminus \{y_0\}\}$ is monochromatic (since $[Y]^2$ is monochromatic). Therefore, X is $H_{B,2} - Infinite$

□

So far, we have a diagram about $H - Finite$ as below.

$$
\begin{tikzcd}
Fin \arrow[d] & \\
H_{B,2} - Fin \arrow[d] \arrow[dr] & \\
H_{B,3} - Fin \arrow[d] & R_2 - Fin \\
H - Fin &
\end{tikzcd}
$$

We have proved that $H_{B,4} - Infinite$ is equivalent to the full Hindman's Theorem, but it is possible that $H_{B,3} - Inifinite$ or $H_{B,2} - Infinite$ is already equivalent to the full Hindman's Theorem. To prove or disprove our guess, we have the following discussion.

Theorem 4.3 *$H_{B,2} - finite$ is not equivalent to $H - finite$.*

Proof: Since H-Infinite is a stronger argument than $H_{B,2}$-Infinite, then clearly, $H_{B,2}$-Finite implies H-Finite. To figure out the implication in the other direction, we use the 1st Fraenkel Model.

Using 1st Fraenkel Model as counterexample to show that H-Finite can not imply H_2-Finite

Let c:$[A]^{<\omega} \to 2$ be an arbitrary map. Since c exists in the model, then there is a finite support F for this map. Define $X = A \setminus F$ (X is an infinite set) and let $c' = c \restriction_{[X]^{<\omega}}$. Since map c is symmetric, then for all permutations that fix F pointwise, $c(x) = c(\pi(x))$ (where $x \in [X]^{<\omega}$). So, under this restriction map, all sets with the same cardinality should be in the same color. Now choose arbitrary point a in X and extend support to be $F' = F \cup \{a\}$. Construct infinite set $Y = \{\{a, b\} | b \in X \setminus \{a\}\}$. Then $F\triangle_{\leqslant 2}(Y) = \{a \cup b | b \in X \setminus \{a\}\} \cup \{c || c| = 2, c \cap a = \emptyset\}$. Clearly that all the sets contained in $F\triangle_{\leqslant 2}(Y)$ have cardinality 2. Therefore $F\triangle_{\leqslant 2}(Y)$ is monochromatic, the set of atoms in the 1st Fraenkel Model is H_2-Infinite.

However, the set of atoms in the 1st Fraenkel Model is $H - Finite$. Prove by contradiction, suppose the set of atoms in the 1st Fraenkel Model is $H - Ininite$, then by definition, there exists injective map f: $\omega \to A$. Fix $n \in \omega$, there exists permutation π and atom a, b, st, $a \in f(n), b \in A \setminus \{F \cup f(n)\}, a \notin F, b \notin F, \pi(a) = b$. Clearly, this π fix support F pointwise, but $\pi(n, f(n)) \neq (n, f(n))$ which leads to a contradiction.

Since the set of atoms in the 1st Fraenkel Model is $H_2 - Infinite$ but $H - Finite$, then $H - Finite$ can not imply $H_2 - Finite$.

□

Relation between $H-Finite$ and $H_{B,3}-Finite$.

Since H-Infinite is a stronger argument than $H_{B,3}$-Infinite, then clearly, $H_{B,3}$-Finite implies H-Finite. To figure out the implication in the other direction, we need either find a model which is $H_{B,3}-Infinite$ but $H-Finite$ as a counterexample, or somehow prove $H_{B,3}-Infinite$ implies $H-Finite$.

We do not have a conclusive result for this relation, we have tried many different permutation models but the set of atoms in all of them are both $H-Finite$ and $H_{B,3}-Finite$. Here I just show one model we have tried.(1st Fraenkel Model).

Theorem 4.4 *Set of atoms in the 1st Fraenkel Model is $H_{B,3}-Finite$.*

Proof: Suppose set of atoms in the 1st Fraenkel Model is $H_3-Infinite$, then by definition, for all coloring maps c: $[A]^{<\omega} \to 2$, there is an infinite set $Y \in [A]^{<\omega}$, st, $F\triangle_{\leqslant 3}(Y)$ is monochromatic. Define a special coloring map $c : [A]^{<\omega} \to 2$, st, $\forall a \in [A]^{<\omega}$,

$$\begin{cases} c(a) = blue & |a| = 0,1(mod 4) \\ c(a) = red & |a| = 2,3(mod 4) \end{cases}$$

Then there exists an infinite set $Y \in [A]^{<\omega}$, st, $F\triangle_{\leqslant 3}(Y)$ is monochromatic. Fix $y \in Y$, st, $y \not\subset F$ (F is support of 1st Fraenkel Model). Rewrite y as $y = b \cup c$, where $b = y \cap F, c = y \setminus F$. Fix one point $x \in c$ and extend support to be $F' = F \cup (c \setminus \{x\})$. Then by the proof of Theorem 4.3, all sets with the same cardinality in the 1st Fraenkel Model are of the same color. Since permutation defined on 1st Fraenkel Model can move x to any other point in $A \setminus F'$, then $Y' = \{(b \cup (c \setminus \{x\})) \cup \{x'\} | x' \in A \setminus F'\}$ is monochromatic and $Y' \subset Y$. Let $|b \cup (c \setminus \{x\})| = k$, then each element of $F\triangle_3(Y')$ has cardinality $k+3$, but each element in Y' has cardinality $k+1$, which means Y' and $F\triangle_3(Y')$ have different color. We arrive at a contradiction. Therefore, set of atoms in the 1st Fraenkel Model is $H_{B,3}-Finite$.

Since set of atoms in the 1st Fraenkel Model is both $H-Finite$ and $H_{B,3}-Finite$, then it can not tell us the relation between $H-Finite$ and $H_{B,3}-Finite$.

Definition 4.6 *A set X is $H_n^{mod\,k}$ if for all coloring maps c: $\bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}} \to 2$, there is an infinite $Y \in \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$, st, $F\triangle_{\leqslant n}(Y)$ is monochromatic.*

Theorem 4.5 *For all $k \in \omega$, $H_{2k}^{mod\,k} - Finite$ is equivalent to $H^{mod\,k} - Finite$.*

Proof: Fix $k \in \omega$. Since $H^{\text{mod k}} - Infinite$ is a stronger statement than $H_{2k}^{\text{mod k}} - Infinite$, then clearly, $H^{\text{mod k}} - Infinite$ implies $H_{2k}^{\text{mod k}} - Infinite$.

For the other direction, suppose X is $H_{2k}^{\text{mod k}} - Infinite$, then for all coloring maps c: $\bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}} \to 2$, there is an infinite $Y \in \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$, st, $F\triangle_{\leqslant 2k}(Y)$ is monochromatic. Define a special coloring map c: $\bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}} \to 2$ which sending every $x \in \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$ to $\lfloor log_2|x| \rfloor$ (mod 2). [*note: $\forall x \in \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$, x is a infinite sequence, each entry in the sequence is some number between* 0 *and* $k-1$, *x only has finitely many nonzero entries, $|x|$ is the number of nonzero entries,write $|x|$ in binary form, coloring map c defined above would give us the furthest position of nonzero digit in $|x|$ modulo* 2] Then there exists an infinite set $Y \subseteq \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$, st, $F\triangle(Y)$ is monochromatic. By lemma, for any cardinality $n \in \omega$, there are only finitely many $y \in Y$, st, $|y| = n$. Construct a set X' with elements $X_1, X_2, ..., X_n, ...$, st, for all i, each set contained in X_i has the same cardinality.X' must be an infinite set, otherwise Y would not be an infinite set. And also, $\bigcup X_i \in \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}, \forall i$. Now, we have a countable, infinite set $\{\bigcup X_i | i \in \omega\} \subseteq \bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$, which means $\bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$ is Dedekind Infinite, X is $H^{\text{mod k}} - Infinite$.

Proof of Lemma (by contradiction) We know that for a fixed $n \in \omega$, there do not exist k elements in $\bigoplus_X \frac{\mathbb{Z}}{k\mathbb{Z}}$ with cardinality n that satisfy any two of these k elements have empty intersection, otherwise, symmetric difference of these k elements will not be of the same color as single element. Thus, for any k elements with the same cardinality, at least two of them must have nonempty intersection. (claim)

Suppose there exists $n \in \omega$, st, there are infinitely many sets in Y with cardinality n. Choose arbitrary set $y \in Y$, st, $|y| = n$. Since we have infinitely many sets with the same cardinality, then by our claim above, all the other elements (infinitely many) should intersect y at some place. Since y is finite, then it has only finitely many subsets. By pigeonhole principle, there should be infinitely many elements with cardinality n intersect y at the same place x_0, $|x_0| = n_0$. Now, collect these elements together to be a new set Y'. Define a coloring map c' on $[Y']^k = \{(y_0,...,y_k) | y_0,...,y_k \in Y'\}$ by the rule that if there exist i, j, st, $(y_i \setminus x_0) \cap (y_j \setminus x_0) \neq \emptyset$, then color the k-tuple blue, otherwise, color the k-tuple red. By Ramsey Theorem, there exist an infinite set Y'', st, $[Y'']^k$ is monochromatic. And furthermore, this color can not be red. Otherwise, we can take $2k$ arbitrary elements from $[Y'']^k$ and construct $S = \{y_1, y_2, ..., y_{2k}\} \subset Y'$, st, every k-tuple from S is colored red. Then $|y_1 \triangle y_2| = |y_3 \triangle y_4| = ... = |y_{2k-1} \triangle y_{2k}| = n - n_0$, and any two of these 2-symmetric difference are disjoint, this contradicts our claim above. Therefore, we have a infinite set $Y''' \subseteq Y'$, st, for any k-tuple from $[Y''']^k$, at least two elements have a bigger intersection than x_0, by pigeonhole principle, we can construct a new set, st, any two elements in this set have bigger intersection than x_0. Repeat those steps above...

Since the set we start with has finite cardinality n, this procedure will end in at most n steps. (since the cardinality of the intersection parts would increase at least 1 at each step, eventually, Ramsey Theorem would guarantee the existence of infinitely many sets with cardinality n, st, for any k-tuple, at least two elements have intersection with cardinality n, which means there is actually only one set. But this result contradicts Ramsey Theorem.) And in each step, instead of requiring the existence of infinitely many red k-tuples, we only need $2k$ elements with disjoint "tails" to arrive at a contradiction. Thus we just need to assume there are $2^n + \underbrace{R(2k, 2^{n-1} + R(2k, \cdots R(2k, 2) \cdots))}_{n-1}$ sets with cardinality n at the beginning of this proof. This is a really big but finite number, so axiom of choice is not involved in this proof.

□