Balanced Triangulations of Sphere Products with Minimal Vertices

Alexander Wang
Advisor: Hailun Zheng

alxwang@umich.edu

August 17, 2018

Abstract

We construct the first balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices for all $d \geq 3$, using a sphere decomposition inspired by handle theory. We then determine some of its properties as well as use computational methods to search for a minimal triangulation.

1 Introduction

Minimal triangulations of manifolds are an important research object in combinatorial and computational topology. In this paper, we study triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with an additional structure called balancedness. A $(d-1)$-dimensional simplicial complex is *balanced* provided that its graph is d-colorable. Many important classes of complexes arise as balanced complexes, such as barycentric subdivisions of regular CW complexes and Coxeter complexes. It is natural to ask the following questions: what is the minimal number of vertices required for a non-balanced or balanced triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$? Is the minimal triangulation unique?

From a result of Brehm and Kühnel [2], we know that a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ has at least $2d - i + 2$ vertices. The problem is also fairly well understood for both the balanced and unbalanced cases of sphere bundles over the circle. A triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $2d + 1$ vertices was constructed by Kühnel [8] in 1986. Later, two groups of researchers, Bagchi and Datta [1] as well as Chestnut, Sapir and Swartz [3], found in 2008 that Kühnel's construction is indeed the unique minimal triangulation for the case when d is odd. Also when d is even, the minimum triangulation requires $2d + 2$ vertices and is not unique. In the balanced case, the result is similar: Klee and Novik [6] provided a balanced triangulation of $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ with $3d$ vertices for odd d and with $3d + 2$ vertices otherwise, and

Zheng [12] showed that the number of vertices for the minimal triangulation is indeed $3d$ for odd d and $3d + 2$ otherwise.

Another research direction on this topic is to find small triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$. The best general result is from [5], where a centrally symmetric triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ with $2d + 2$ vertices is constructed as a subcomplex of the octahedral d-sphere. In addition, the minimal triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \leq 7$ as well as the minimal triangulation of $\mathbb{S}^3 \times \mathbb{S}^3$ are obtained using BISTELLAR, a program written by Lutz [9]. However, as of yet, no balanced triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-1}$ for $2 \leq i \leq d-3$ exist in literature.

In this paper, we construct a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices for all $d \geq 3$. The intuition behind our construction is from a sphere decomposition inspired by handle theory. Recently, Izmestiev, Klee, and Novik [4] proved that any two balanced PL homeomorphic closed combinatorial manifolds can be connected using a sequence of cross-flips. In particular, given a balanced triangulation of a manifold, this allows us to computationally search for a minimal balanced triangulation. The ongoing project is to develop a program with Lorenzo Venturello which applies cross-flips to search for the minimal balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for small d.

The paper is structured as follows. In section 2, we review the basics of simplicial complexes, balanced triangulations, and other definitions that may be relevant. In section 3, we present our main construction and show that it does indeed triangulate the desired manifold. We then provide the face enumeration as well as the automorphism group. In Appendix A, we briefly discuss the computational work done towards finding a minimal balanced triangulation.

2 Preliminaries

A *simplicial complex* Δ with vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximum dimension of the faces of Δ. A face $\sigma \in \Delta$ is said to be a *facet* provided that it is a face which is maximal with respect to inclusion. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension. If Δ is $(d-1)$-dimensional and $-1 \leq i \leq d-1$, then the *f-number* $f_i = f_i(\Delta)$ denotes the number of i-dimensional faces of Δ. The *star* and *link* of a face σ in Δ is defined as follows:

$$\operatorname{st}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\} \qquad \operatorname{lk}_\Delta \sigma := \{\tau \in \operatorname{st}_\Delta \sigma : \tau \cap \sigma = \emptyset\}$$

When the context is clear, we may simply denote the star and link of σ as $\operatorname{st}(\sigma)$ and $\operatorname{lk}(\sigma)$. We also define the *restriction* of Δ to a vertex set W as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. A subcomplex $\Omega \subset \Delta$ is said to be *induced* (sometimes *full*) provided that for all faces $F \in \Delta$, if every vertex $v \in F$ is a vertex of Ω, then F is a face in Ω.The *i-skeleton* of a simplicial complex Δ is the subcomplex containing all faces of Δ which have dimension at most i. In particular, the 1-skeleton of Δ is the graph of Δ.

A simplicial complex Δ is a *simplicial manifold*, or equivalently, a triangulated manifold, if the geometric realization of Δ is homeomorphic to a manifold. The boundary complex of a simplicial d-ball is a simplicial $(d-1)$-sphere. We write $\tilde{H}_*(\Delta;\mathbf{k})$ to denote the reduced homology of Δ with coefficients in $\mathbb{Z}$.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d-colorable; that is, there exists a coloring map $\kappa : V \to [d]$ such that $\kappa(x) \neq \kappa(y)$ for all edges $\{x, y\} \in \Delta$. Here $[d] = \{1, 2, \cdots, d\}$ denotes the set of colors.

Let ∂C_d^* be the boundary complex of the d-crosspolytope. It is a well known result that this is a balanced vertex-minimal triangulation of the $(d-1)$-sphere. Label the vertex set of ∂C_d^* as $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ such that x_i, y_i form a pair of antipodal vertices for all i. Every facet of ∂C_d^* can be written in the form $u_1u_2 \ldots u_d$, where each $u_i \in \{x, y\}$. We say a facet has a switch at position i if u_i and u_{i+1} have different labels. Let $B(i, d)$ be the subcomplex of ∂C_d^* that contains all facets with at most i switches. For example, $B(0, d)$ consists of the two disjoint facets $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$. The following lemma follows directly from Theorem 1.2 in [5]. Here, D^i denotes the i-dimensional unit ball.

Lemma 2.1. *For $0 \leq i < d-1$, the complex $B(i, d)$ satisfies the following properties:*

1. *$B(i, d)$ contains the entire i-skeleton of ∂C_d^* as a subcomplex.*
2. *The boundary of $B(i, d)$ is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.*
3. *$B(i, d)$ is a balanced cs combinatorial manifold whose integral (co)homology groups coincide with those of $\mathbb{S}^i$. Also, $B(0, d) \cong D^{d-1} \times \mathbb{S}^0$ and $B(1, d) \cong D^{d-2} \times \mathbb{S}^1$.*
4. *The complement of $B(i, d)$ in ∂C_d^* is simplicially isomorphic to $B(d-i-2, d)$.*
5. *B(i,d) admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_d$ if i is even and of $\mathcal{D}_{2d}$ if i is odd.*

3 Balanced Triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$

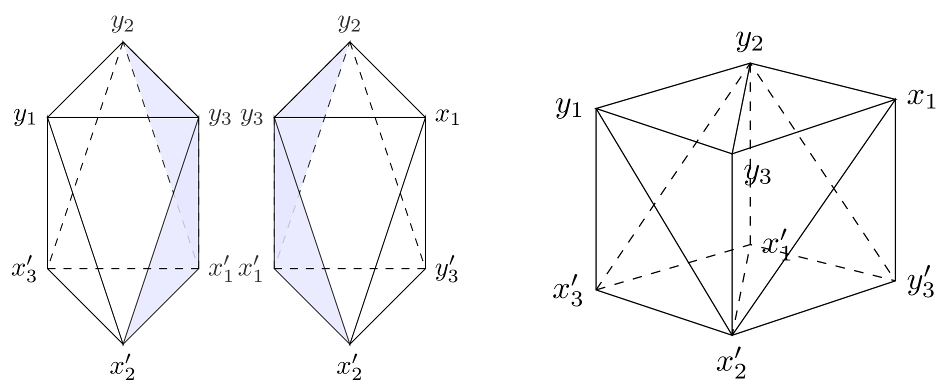
In this section, we present our main construction for a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. The geometric intuition of our construction comes from handle theory. The sphere $\mathbb{S}^{d-1}$ admits the following decomposition:

$$\mathbb{S}^{d-1} = \partial D^d = \partial(D^2 \times D^{d-2}) = (\partial D^2 \times D^{d-2}) \cup (D^2 \times \partial D^{d-2}) = (\mathbb{S}^1 \times D^{d-2}) \cup (D^2 \times \mathbb{S}^{d-3}).$$

Let S be a triangulated $(d-1)$-sphere that has the decomposition $S = B_1 \cup_{\partial B_1 = \partial B_2} B_2$, where $B_1 \cong \mathbb{S}^1 \times D^{d-2}$, $B_2 \cong D^2 \times \mathbb{S}^{d-3}$, and $\partial B_1 \cong \partial B_2 \cong \mathbb{S}^1 \times \mathbb{S}^{d-3}$ (where D^i denotes the i-dimensional unit ball). Note that $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ admits the decomposition into $(D^2 \times \mathbb{S}^{d-3}) \cup (D^2 \times \mathbb{S}^{d-3}) \cong B_2 \cup B_2$. Then, from S we can form a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-2}$ in the following way: take two copies of B_2 and denote them as B_2 and B_2'. If ∂B_2 is an induced subcomplex in B_2, then we glue B_2 and B_2' along their boundaries. The resulting complex is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. However, if ∂B_2 is not an induced subcomplex of B_2, then usually we cannot glue B_2 and B'_2 by identifying their boundaries directly and still obtain a triangulated manifold. An alternative method is to find a complex $N \cong \partial B_2 \times D^1$ with $\partial N = \partial B_2 \cup \partial B'_2$ so that N serves as a tubular neighborhood of both ∂B_2 and $\partial B'_2$. Finally the complex $B_2 \cup N \cup B'_2$ is a triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$.

Our approach of constructing a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ is by finding suitable balanced candidates of B_2 and N as described above. We begin by defining a variation of the usual connected sum.

Definition 3.1. Consider (Γ_1, σ_1) and (Γ_2, σ_1), where Γ_i is the boundary complex of the d-cross-polytope, and σ_i is a fixed facet of Γ_i. Let κ be the coloring map on $\Gamma_1 \cup \Gamma_2$. If e_i is an edge in Γ_i but not in $\pm\sigma_i$ and $\kappa(e_1) = \kappa(e_2)$, then the $\diamond$-connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ is obtained by deleting e_i from Γ_i, and gluing $\Gamma_1 - e_1$ with $\Gamma_2 - e_2$ by identifying the vertices of the same color in $\partial\, \mathrm{st}_{\Gamma_1} e_1$ and $\partial\, \mathrm{st}_{\Gamma_2} e_2$.



(a) Γ_1 and Γ_2; here $\sigma_1 = \{y_1, y_2, y_3\}$ and $\sigma_2 = \{x_1, y_2, y_3\}$

(b) $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$

Figure 1: The $\diamond$-connected sum $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$: delete the edge $\{y_3 x'_1\}$ in both Γ_1 and Γ_2, then glue Γ_1 and Γ_2 along the 4-cycle (y_3, x'_2, x'_1, y_2).

The following properties of the $\diamond$-connected sum justify the notation $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ in the definition.

Property 3.2. *Let Γ_1 and Γ_2 be two d-crosspolytopes. Furthermore, Γ_1 has antipodal facets $\sigma_1 = \{x_1, \dots, x_d\}$, $-\sigma_1 = \{y_1, \dots, y_d\}$, and Γ_2 has antipodal facets $\sigma_2 = \{x_{d+1}, \dots, x_{2d}\}$, $-\sigma_2 = \{y_{d+1}, \dots, y_{2d}\}$. Then $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ satisfies the following properties:*

1. *The complex is a balanced triangulation of $\mathbb{S}^{d-1}$.*

2. *The restriction of $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ to $V(\sigma_1) \cup V(\sigma_2)$ is the usual connected sum of simplices $\sigma_1 \# \sigma_2$.*

3. *The link of every edge $e = \{x_i, y_j\}$ in $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ is the boundary complex of a $(d-2)$-crosspolytope.*

Proof: Part 1 is clear from the construction. For part 2, if $e_1 = \{x_i, y_j\}$ in Γ_1, then the link $\mathrm{lk}_{\Gamma_1} e_1$ is the boundary of a $(d-2)$-cross-polytope containing the antipodal facets $\sigma_1 \backslash \{x_i, x_j\}$ and $(-\sigma_1)\backslash\{y_i, y_j\}$. Similarly, for $e_2 = \{x_k, y_l\}$ in Γ_2, the link $\mathrm{lk}_{\Gamma_2} e_2$ has antipodal facets $\sigma_2\backslash\{x_k, x_l\}$ and $(-\sigma_2)\backslash\{y_k, y_l\}$. Hence the restriction of $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ to $V(\sigma_1) \cup V(\sigma_2)$ is obtained by taking the union of σ_1 and σ_2 and identifying $\sigma_1\backslash\{x_j\}$ with $\sigma_2\backslash\{x_l\}$. In this manner, we get the connected sum $\sigma_1 \# \sigma_2$.

For part 3, let Δ denote the boundary complex on which Γ_1 and Γ_2 are glued together. If $e \notin \Delta$, there is nothing to prove. Otherwise, assume without loss of generality that $e = \{x_1, y_2\}$ and the edge $e' = \{x_1, y_3\}$ is deleted from Γ_1 to form $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$. Then, $\mathrm{lk}_{\mathrm{st}_{e'} \Gamma_1} e = \{y_3\} * \Sigma$, where Σ is the boundary of the crosspolytope on vertices $\{x_4, y_4, \ldots, x_d, y_d\}$. Hence, by construction, the link of e in $(\Gamma_1 \# \Gamma_2, \sigma_1 \# \sigma_2)$ must be the suspension of Σ; here the suspension vertices are the antipodal vertices of y_3 in Γ_1 and Γ_2 respectively. $\square$

The above properties ensure that it is possible to take the $\diamond$-connected sum inductively. To form $(\Gamma_1 \# \ldots \# \Gamma_k, \sigma_1 \# \ldots \# \sigma_k)$ from $(\Gamma_1 \# \ldots \# \Gamma_{k-1}, \sigma_1 \# \ldots \# \sigma_{k-1})$ and (Γ_k, σ_k), we take an edge $e_1 \in (\Gamma_1 \# \ldots \# \Gamma_{k-1}, \sigma_1 \# \ldots \# \sigma_{k-1})$ but not in $\sigma_1 \# \ldots \sigma_{k-1}$, then take an edge $e_2 \in \Gamma_k \backslash \sigma_k$ so that $\kappa(e_1) = \kappa(e_2)$, and then construct as in Definition 3.1.

Recall that if Γ is a pure simplicial complex, then as long as there exist two facets F and F' on Γ and a map $\phi : F \to F'$ so that v and $\phi(v)$ do not have a common neighbor for every $v \in F$, then we can remove F, F' and identify ∂F with $\partial F'$ to get Γ^ϕ. This is called a handle addition. Similarly, assume that there are two edges e_1 and e_2 of the same color in $(\Gamma_1 \# \ldots \# \Gamma_k, \sigma_1 \# \ldots \# \sigma_k)$ but not in $A := \sigma_1 \# \sigma_2 \ldots \# \sigma_k$ or $-A$. Note that $\mathrm{st}(e_i)$ is a cross-polytope with antipodal facets $\mathrm{st}(e_i)[V(A)]$ and $\mathrm{st}(e_i)[V(-A)]$ for $i = 1, 2$. If the identification maps $\phi : \mathrm{st}(e_1)[V(A)] \to \mathrm{st}(e_2)[V(A)]$ and $\phi' : \mathrm{st}(e_1)[V(-A)] \to \mathrm{st}(e_2)[V(-A)]$ are well-defined, then the maps ϕ and ϕ' naturally extend to a map $\bar{\phi} : \mathrm{st}(e) \to \mathrm{st}(e')$ if for every $v \in \mathrm{st}(e)$, v and $\phi(v)$(or $\phi'(v)$) do not have a common neighbor. In this way we obtain a balanced simplicial complex $((\Gamma_1 \# \Gamma_2 \ldots \# \Gamma_k)^{\bar{\phi}}, (\sigma_1 \# \sigma_2 \ldots \# \sigma_k)^\phi)$ by removing e, e' and identifying $\mathrm{lk}(e)$ with $\bar{\phi}(\mathrm{lk}(e)) = \mathrm{lk}(e')$. We call this the $\diamond$-handle addition. Note that as long as the handle addition is well-defined, $f_0((\Gamma_1 \# \Gamma_2 \ldots \# \Gamma_k)^{\bar{\phi}}) = 2 f_0((\sigma_1 \# \sigma_2 \ldots \# \sigma_k)^\phi) = 2k$ regardless of the dimension of Γ_i.

We are now ready to construct a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ with $4d$ vertices. We will write $\Gamma_1 \# \Gamma_2$ to denote the $\diamond$-connected sum if σ_1 and σ_2 are clear from the context. Also, to simplify notation, we will sometimes write $x_1 \ldots x_m$ to denote the face $\{x_1, \ldots, x_m\}$.

Example 3.3. Let $d \geq 3$. Take two d-crosspolytopes P and P'. The vertex sets of P and P' are $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and $\{x'_1, \ldots, x'_d, y'_1, \ldots, y'_d\}$ respectively. We let $\sigma_i = x_1 \ldots x_i y_{i+1} \ldots y_d$ for $1 \leq i \leq d$ and let $\sigma_i = y_1 \ldots y_i x_{i+1} \ldots x_d$ for $d+1 \leq i \leq 2d$. Then the complex $\Delta_1 := \cup_{i=1}^{2d} \sigma_i$ is exactly $B(1, d)$. We further partition the boundary of P as $\partial P = \Delta_1 \cup_{\partial \Delta_1} \Delta_2$. By Lemma 1.1, $\Delta_2 \cong B(d-3, d)$ and $\Delta_1 \cap \Delta_2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{d-3}$.

Next, define a simplicial map $f : \partial P \to \partial P'$ induced by the following bijection on the vertex sets:

$$x_i \mapsto x'_{i+1},\ y_i \mapsto y'_{i+1} \text{ for } 1 \leq i \leq d-1;\ x_d \mapsto y'_1,\ y_d \mapsto x'_1.$$

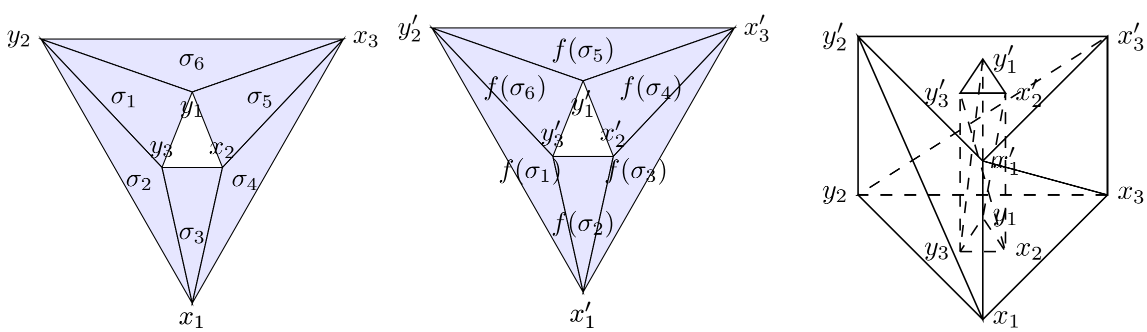
By Lemma 2.1, the complex Δ_1 admits a vertex-transitive action by the dihedral group $\mathcal{D}_{2d}$ of order $4d$, where a generator is given by the map we have chosen (Theorem 1.2 of [5, page 6]). Hence f is a simplicial isomorphism and $f(\Delta_1) \cong B(1, d)$. For each i, there is a unique d-cross-polytope Γ_i containing σ_i and $f(\sigma_i)$ as antipodal facets. Next, we check that we can take the $\Diamond$-connected sum of Γ_i and Γ_{i+1} inductively. Without loss of generality, assume that $1 \leq i \leq d$; otherwise, we can relabel by switching x and y. Note that for $i \leq d-2$,

$$\sigma_i = x_1 x_2 \ldots x_i y_{i+1} y_{i+2} \ldots y_d, \quad \sigma_{i+1} = x_1 x_2 \ldots x_{i+1} y_{i+2} y_{i+3} \ldots y_d,$$

$$\text{and} \quad f(\sigma_i) = x'_2 x'_3 \ldots x'_{i+1} y'_{i+2} y'_{i+3} \ldots y'_d y'_1, \quad f(\sigma_{i+1}) = x'_2 x'_3 \ldots x'_{i+2} y'_{i+3} y'_{i+4} \ldots y'_d y'_1.$$

Hence, $\sigma_i \cap \sigma_{i+1} = x_1 x_2 \ldots x_i y_{i+2} \ldots y_d$ and $f(\sigma_i) \cap f(\sigma_{i+1}) = x'_2 x'_3 \ldots x'_{i+1} y'_{i+3} \ldots y'_d y'_1$. The missing indices are $i+1$ and $i+2$ respectively, so we let $e_i = x'_{i+1} y_{i+2}$. It follows that $\Gamma_i \cap \Gamma_{i+1} = \mathrm{st}_{\Gamma_i} e_i = \mathrm{st}_{\Gamma_{i+1}} e_i$ and hence the $\Diamond$-connected sum is well defined. Similarly, $\Gamma_{d-1} \cap \Gamma_d = \mathrm{st}_{\Gamma_d}\{x'_d, x_1\}$ and $\Gamma_d \cap \Gamma_{d+1} = \mathrm{st}_{\Gamma_d}\{y'_1, x_2\}$. Inductively, we form a complex $\Gamma = ((\Gamma_1 \# \Gamma_2 \ldots \Gamma_{2d})^{\phi}, \Delta_1)$ which contains Δ_1 and $f(\Delta_1)$ as subcomplexes.

We partition Γ as $\Gamma = \Delta_1 \cup f(\Delta_1) \cup N$, so that $N \cap \Delta_1 = \partial\Delta_1$ and $N \cap f(\Delta_1) = \partial f(\Delta_1)$. N is then the tubular neighborhood that we would like to construct. Finally, let $\Sigma = \Delta_2 \cup_{\partial\Delta_1} N \cup_{\partial f(\Delta_1)} f(\Delta_2)$. (This is well defined as by Lemma 2.1, $\partial\Delta_1 \cong \partial\Delta_2$.)



(a) Δ_1 (b) $f(\Delta_1)$ (c) Γ

Figure 2: The complexes Δ_1 and $f(\Delta_1)$ when $d = 3$, and the resulting Γ constructed using the previously described sequence of connected sums.

In the specific case of $d = 3$, we have the manifold $\mathbb{S}^2 \times \mathbb{S}^0$, which consists of two disjoint spheres. In this case, the construction gives the boundary of two 3-cross-polytopes, which is the minimal triangulation. For arbitrary d, we will need the following theorem from [7] to show that Σ triangulates the desired manifold.

Theorem 3.4. *Let M be a simply connected codimension-1 submanifold of $\mathbb{S}^{d-1}$, where $d \geq 6$. If M has the homology of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ and $1 < i \leq d/2 - 1$, then M is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.*

Next we check that Σ satisfies all the conditions as described in the above theorem.

Lemma 3.5. *The complex Σ given in Example 3.3 is simply connected for $d \geq 5$.*

Proof: Since $B(d-3,d)$ contains the 2-skeleton of ∂C_d^*, it follows that both Δ_2 and $f(\Delta_2)$ are simply connected. Then $A := \Delta_2 \cup \Gamma'$ and $B := \Gamma' \cup f(\Delta_2)$ are also simply connected. Write $\Sigma = \mathrm{int}(A) \cup \mathrm{int}(B)$. It is easy to see that both Σ and $A \cap B = \mathrm{int}(\Gamma')$ are path connected. The result follows since the union of two simply connected open subsets with path-connected intersection is simply connected. $\square$

Proposition 3.6. *The complex Σ constructed above is a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 5$.*

Proof: By the theorem and the lemma above, it suffices to check that Σ has the same homology as $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ for $d \geq 4$. Applying the Mayer-Vietoris sequence on the triple $(\Delta_2 \cup f(\Delta_2), \Gamma', \Sigma)$, for all i, we have:

$$\cdots \to H_{i+1}(\Sigma) \to H_i(\partial\Delta_2 \cup \partial f(\Delta_2)) \to H_i(\Gamma') \oplus H_i(\Delta_2 \cup f(\Delta_2)) \to H_i(\Sigma) \to \cdots .$$

Since $\Gamma' \cong \partial\Delta_2 \times [0,1]$, $\frac{1}{2}\beta_i(\partial\Delta_2 \cup \partial f(\Delta_2)) = \beta_i(\Gamma') = 1$ for $i = 1, d-3, d-2$, and zero otherwise. Also by the lemma, $\beta_i(\Delta_2 \cup f(\Delta_2)) = 2\beta_i(\Delta_2) = 2$ for $i = d-3$, and zero otherwise. Hence, we obtain that

$$0 \to H_{d-1}(\Sigma) \to H_{d-2}(\partial\Delta_2 \cup \partial f(\Delta_2)) \to H_{d-2}(\Gamma') \oplus H_{d-2}(\Delta_2 \cup f(\Delta_2)) = 0,$$

which implies that $\beta_{d-1}(\Sigma) = 1$. Also

$$0 \to H_{d-2}(\Sigma) \to H_{d-3}(\partial\Delta_1 \cup \partial f(\Delta_1)) \to^i H_{d-3}(\Gamma') \oplus H_{d-3}(\Delta_2 \cup f(\Delta_2)) \to H_{d-3}(\Sigma) \to 0.$$

Since the map i must be injective, it follows that $\beta_{d-2}(\Sigma) = 0$ and $\beta_{d-3}(\Sigma) = 1$. Finally,

$$0 \to H_2(\Sigma) \to H_1(\partial\Delta_2 \cup \partial f(\Delta_2)) \to^{i'} H_1(\Gamma') \to H_1(\Sigma) \to 0.$$

Again, the map i' is surjective, which forces $\beta_2(\Sigma) = 1$ and $\beta_1(\Sigma) = 0$. It is clear to see that the other Betti numbers of Σ are zero. Finally, the balancedness of Σ follows from the construction. $\square$

Note that the construction in Figure 2 is for the case $d = 3$, and forms a balanced triangulation of $\mathbb{S}^2 \times \mathbb{S}^0$ which is indeed minimal. The case of $d = 4$, where Σ triangulates $\mathbb{S}^3 \times \mathbb{S}^1$, can be checked computationally. However, from [12], we know that this triangulation of 16 vertices is not minimal, as a minimal triangulation of $\mathbb{S}^1 \times \mathbb{S}^3$ contains 14 vertices.

We list several properties of Σ:

Proposition 3.7. *For $d \geq 4$, let Σ be the triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$ as constructed in Example 3.3. Then, Σ has the following face numbers.*

- $f_0(\Sigma) = 4d$
- $f_1(\Sigma) = 4d(2d-3)$
- $f_{d-1}(\Sigma) = (d+2)2^d - 8d.$

Proof: The complex Σ has $4d$ vertices since $f_0(\Sigma) = f_0(\Delta_2) + f_0(f(\Delta_2))$. By the construction, there are $2d$ edges $e_1 = \{x'_2, y_3\}, \dots, e_{2d} = \{x'_1, y_2\}$ deleted from the cross-polytopes $\Gamma_1, \Gamma_2, \dots, \Gamma_{2d}$ to form Γ. Each Γ_i and $\mathrm{lk}_{\Gamma_i} e_i$ are $(d-1)$-dimensional and $(d-3)$-dimensional octahedral spheres respectively, so we have that $f_1(\Gamma_i) = 2d(d-1)$ and $f_1(\mathrm{st}_{\Gamma_i} e_i) = 2(d-1)(d-2)+1$. Thus,

$$\begin{aligned} f_1(\Gamma_1 \# \Gamma_2 \dots \Gamma_{2d})^{\phi}) &= \sum_{i=1}^{2d} \Big(f_1(\Gamma_i) - f_1(\mathrm{st}_{\Gamma_i} e_i) - 1 \Big) \\ &= 4d^2(d-1) - 4d(d-1)(d-2) - 4d \\ &= 4d(2d-3). \end{aligned}$$

It follows from $f_1(\Delta_1) = f_1(\Delta_2)$ that $f_1(\Sigma) = f_1(\Gamma) = 4d(2d-3)$. Similarly, since the facets in each $\mathrm{st}_{\Gamma_i} e_i$ are disjoint,

$$\begin{aligned} f_{d-1}(\Gamma) &= \sum_{i=1}^{2d} \Big(f_{d-1}(\Gamma_i) - 2 f_{d-1}(\mathrm{st}_{\Gamma_i} e_i) \Big) \\ &= 2d(2^d - 2^{d-1}) \\ &= d2^d. \end{aligned}$$

It follows that $f_{d-1}(\Sigma) = f_{d-1}(\Gamma) - 2f_{d-1}(\Delta_1) + 2f_{d-1}(\Delta_2) = d2^d - 4d + (2^{d+1} - 4d) = (d+2)2^d - 8d$. $\square$

Note that the construction for $d = 3$ satisfies the same face number relations, with the exception of f_1. In this case, there are 24 edges rather than 36, since edges are lost when we replace Δ_1 with Δ_2. This does not happen when $d \geq 4$ as by Theorem 1.2 in [5], we have that for $d \geq 4$, $B(d-3, d)$ contains all edges in the cross-polytope.

Proposition 3.8. *For $d \geq 4$, $\mathrm{Aut}(\Sigma)$ admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_{2d}$.*

Proof: A simplicial map g on the simplicial complex Σ is an isomorphism if it gives a bijection on the facets of Σ. A necessary condition for g to be an automorphism is that it sends the missing edges in Σ to missing edges in Δ. Define the following three permutations, modified from the proof of Theorem 1.2(b) in [5]:

- D maps x_j to y_j, y_j to x_j, x'_j to y'_j and y'_j to x'_j.
- E' maps x_j to x'_{d-j+1}, y_j to y'_{d-j+1}, x'_j to x_{d-j+1} and y'_j to y_{d-j+1}.
- R' maps x_d to y_1, y_d to x_1, x_j to x_{j+1}, y_j to y_{j+1}, and similarly for x'_j and y'_j.

The maps D and E' have order 2, whereas R' has order $2d$. Also note that E' is the permutation E from [5] composed with a switching between the prime and nonprime vertices. We know that of the edges in Σ, the only missing edges are edges between antipodal vertices in Γ_i and the edges deleted when we join Γ_i and Γ_{i+1}; they are $\{x_i' y_{i+1}\}, \{y_i' x_{i+1}\}$ for $1 \leq i \leq d-1$ together with $\{x_d' x_1\}, \{y_d' y_1\}$. It is straightforward to check that D, E', and R' are bijections on the vertices of Σ, and additionally fix setwise the set of missing edges. Since $E'R' = R'^{-1}E'^{-1}$, E' and R' generate $\mathcal{D}_{2d}$, and since D commutes with both E' and R', we have that the three together generate $\mathbb{Z}_2 \times \mathcal{D}_{2d}$.

By Theorem 1.2(b) of [5], we have that facets in Δ_1 and $f(\Delta_1)$, as well as those in Δ_2 and $f(\Delta_2)$, are mapped bijectively by $g = D, E'$ or R'. Therefore, it suffices to show that the facets in the tubular neighborhood N are mapped bijectively. Note that any facet F in N must also be contained in some Γ_i. Therefore, the only way in which $g(F)$ could not be a facet of Σ is if $g(F)$ is in the star of an edge which is deleted. However, as we observed above, g gives bijection on the missing edges of Σ, i.e., $g(F) \in \mathrm{st}(g(e))$ for some missing edge e if and only if $g \in \mathrm{st}(e)$. Hence g is a bijection on the facets of Σ, and so $g \in \mathrm{Aut}(\Sigma)$. $\square$

For the case of $d = 3$, we can directly compute the automorphism group. The construction consists of two disjoint octahedron, with an additional possibility of interchanging the two octahedron. Then, the automorphism group is given by $\mathbb{Z}_2 \times O_h \times O_h$, where O_h denotes the octahedral group of order 48. This group has order $2 \cdot 48^2 = 4608$.

4 Acknowledgements

The authors are partially supported by the Mathematics REU program at the University of Michigan during Summer 2018. We would also like to thank Lorenzo Venturello for his contributions to the computational aspect of this project.

A Appendix: Computational Results

This appendix will briefly discuss our computation results. We developed a Python/Sage program to produce our $4d$ vertex triangulation of $\mathbb{S}^2 \times \mathbb{S}^{d-3}$. In addition, working with Lorenzo Venturello, we created an analog of the BISTELLAR program for balanced simplicial complexes to attempt to reduce the number of vertices of a given triangulation. The program uses a simulated annealing approach, much like the method BISTELLAR uses. However, the complexity of finding shellable subcomplexes in the d-cross-polytope grows exponentially with d, and so the program is highly inefficient for $d > 4$. In addition, cross-flip sequences connecting two different triangulations tend to be much more delicate and structured, and so simulated annealing works poorly on balanced complexes which cannot be immediately reduced by a cross-flip. A few heuristics could be used to improve the search, including attempting to minimize the ratio f_2/f_1 or the sum (or square-sum) of the degree of the vertices in the 1-skeleton of the triangulation.

References

[1] B. Bagchi and B. Datta: Minimal Triangulations of sphere bundles over the Circle, *J. Combin. Theory Ser. A*, **115** (2008), 737-752.

[2] U. Brehm and W. Kühnel: Combinatorial manifolds with few vertices, *Topology* **26** (1987), 465-473.

[3] J. Chestnut, J. Sapir, and E. Swartz: Enumerative properties of triangulations of spherical bundles over S^1, *European J. Combin.*, **29** (2008), 662-671.

[4] I. Izmestiev, S. Klee, and I. Novik: Simplicial moves on balanced complexes, *Advances in Mathematics* **320C** (2017), 82-114.

[5] S. Klee and I. Novik: Centrally symmetric manifolds with few vertices, *Advances in Mathematics* **229** (2012), 487-500.

[6] S. Klee and I. Novik: Lower Bound Theorems and a Generalized Lower Bound Conjecture for balanced simplicial complexes, *Mathematika* **62** (2016), 441-477.

[7] M. Kreck: An inverse to the Poincaré conjecture, *Fretschrift: Erich Lamprecht*, Arch. Math. (Basel) **77** (2001), 98-106.

[8] W. Kühnel: Higher-dimensional analogues of Czászárs torus, *Results Math.* **9**(1986), 95-106.

[9] F. Lutz: The Manifold Page. `http://page.math.tu-berlin.de/~lutz/stellar/`.

[10] I. Novik and E. Swartz: Socles of Buchsbaum modules, complexes and posets, *Advances in Mathematics* **222** (2009), 2059-2084.

[11] C. Rourke and B. Sanderson: *Introduction to Piecewise-Linear Topology*, Berlin Heidelberg New York: Springer-Verlag, 1982.

[12] H. Zheng: Minimal balanced triangulations of sphere bundles over the circle, *SIAM J. Disc. Math.* **222** (2017), 1259-1268.