THE HARRISON - SHEPP EQUATION AND SOME OF ITS OFFSPRING

Talk by

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Abstract

In a pioneering work from 1981, HARRISON and SHEPP provided a stochastic integral equation characterizing the skew Brownian Motion of ITÔ & McKEAN (1963, 1965). We provide similar characterizations for skew-unfoldings of continuous semimartingales, and for a class of planar processes with a roundhouse singularity at the origin, that we call "WALSH semimartingales" and which include the celebrated WALSH Brownian motion as a special case. Armed with this description, and with an associated stochastic calculus that we develop, we formulate and solve problems of optimal control with discretionary stopping for such WALSH semimartingales.

Let us suppose we are given a continuous semimartingale

$$\underbrace{\mathcal{U} = M + A}$$

M: continuous local martingale, $M(0)=0$

A: process of finite variation, $A(0)=0$

There are two ways to reflect this process about the origin:

(i) CONVENTIONAL Reflection ("Folding"):

$$R(t) = |\mathcal{U}(t)|, \quad 0 \le t < \infty.$$

(ii) SKOROHOD Reflection

$$S(t) = \mathcal{U}(t) + \max_{0\le s\le t}\big(-\mathcal{U}(s)\big), \quad 0 \le t < \infty.$$

We know from P. LÉVY's Theorem that, if $\mathcal{U}$ is Brownian motion, the two reflections R, S have the same distribution:

THEY ARE BOTH REFLECTING BROWNIAN MOTIONS.

IS THERE A WAY TO INTERPRET THE SKOROHOD REFLECTION ALSO AS A FOLDING ?

In other words, to find a suitable continuous semimartingale X, related to $\mathcal{U}$ in an appropriate manner, whose conventional reflection (folding) is $\mathcal{S}$, the SKOROHOD reflection of $\mathcal{U}$, i.e.,

$$|X| = \mathcal{S} \ ?$$

This amounts, effectively, to an "unfolding" of $\mathcal{S}$.

And if such an unfolding X exists, how is it related to $\mathcal{U}$?

THEOREM : SKEW-UNFOLDING OF S

Fix a constant $\alpha \in (0,1)$. On a suitable enlargement of the original probability space, with a measure-preserving transformation, there exist a continuous semimartingale X with the properties

$$|X| = S \qquad \text{(X is an unfolding of S)}$$

$$L^X = \alpha L^S \qquad \text{(thinning of local time)}$$

$$X = \int_0^{\cdot} \overline{\mathrm{sgn}}(X(t))\, dU(t) + \frac{2\alpha - 1}{\alpha} L^X . \qquad \text{(dynamics)}$$

□

Here, for a continuous semimartingale Y, we denote by

$$L^Y \triangleq \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} \mathbb{1}_{[0,\varepsilon)}(Y(t))\, d\langle Y \rangle(t) = Y^+ - \int_0^{\cdot} \mathbb{1}_{\{Y(t)>0\}}\, dY(t)$$

the "local time from the right"; for $Y \geq 0$, this becomes

$$L^Y = \int_0^{\cdot} \mathbb{1}_{\{Y(t)=0\}}\, dY(t).$$

Also,

$$\overline{\mathrm{sgn}}(x) \triangleq \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}, \qquad \mathrm{sgn}(x) = \begin{cases} 1, & x>0 \\ -1, & x \leq 0 \end{cases}.$$

Remark: Under the additional condition

$$\int_0^\infty \mathbb{1}_{\{S(t)=0\}}\, dA(t) = 0$$

we have the filtration comparisons

$$\mathcal{F}^{|X|}(t) = \mathcal{F}^{\mathcal{U}}(t) \subsetneq \mathcal{F}^{X}(t)$$

for $0 \le t < \infty$. In fact,

$$\mathcal{U} = \int_0^{\cdot} \operatorname{sgn}(X(t))\, dX(t) + \frac{2\alpha-1}{\alpha} L^X$$

gives a recipe for constructing the paths of $\mathcal{U}$ from those of X.

But the reverse is not possible.

IDEA of PROOF: Take the excursions of S away from the origin; enumerate them as $\mathcal{C}_k$, $k \in \mathbb{N}$; and for each of them toss a coin

$$\mathbb{P}(\xi_k = +1) = \alpha, \quad \mathbb{P}(\xi_k = -1) = 1-\alpha$$

with $\mathbb{E}(\xi_k) = 2\alpha - 1$, independently from one excursion to all the others, and independently of $\mathcal{U}$ (thus of S).

If the toss comes up heads, we keep the excursion as is; if it comes up tails, we flip the excursion about the origin.

Thus, with

$$\mathcal{Z} = \{t \geq 0 : S(t) = 0\}, \quad [0,\infty) \setminus \mathcal{Z} = \bigcup_{k \in \mathbb{N}} \mathcal{C}_k$$

and setting formally $\mathcal{C}_0 = \mathcal{Z}$, $\xi_0 = 0$, we define

$$H(t) \triangleq \sum_{k \in \mathbb{N}_0} \xi_k \mathbb{1}_{\mathcal{C}_k}(t), \qquad X(t) = H(t) S(t), \quad 0 \leq t < \infty.$$

All our processes are now adapted to the enlarged (augmented) filtration

$$\tilde{\mathcal{F}}(t) = \mathcal{F}(t) \vee \mathcal{F}^{Z}(t) ;$$

and it can be shown that M remains a local martingale under this enlargement.

This implies, that our semimartingales remain semimartingales.

Now the process H takes values in $\{-1, 0, 1\}$; thus $|X| = S$. It can also be seen that

$$X = HS = \int_0^{\cdot} H(t)\, dS(t) + (2\alpha - 1) L^S .$$

Once this has been established, the remaining claims can follow, after a bit of work.

But how about

$$HS = \int_0^{\cdot} H(t)\,dS(t) + (2\alpha-1)L^S \qquad ?$$

in particular, what is the local time L^S doing here?

To see this, we appoximate the "rough" process H by piecewise constant processes

$$H^\varepsilon(t) = \sum_{l \in \mathbb{N}_0} H(t)\,\mathbb{1}_{(\tau^\varepsilon_{2l+1},\,\tau^\varepsilon_{2l+2}]}(t).$$

The product rule gives for them

$$H^\varepsilon(T)S(T) = \int_0^T H^\varepsilon(t)\,dS(t) + \int_0^T S(t)\,dH^\varepsilon(t)$$

$$\downarrow \quad \left(\begin{smallmatrix}\text{in probability}\\ \text{as } \varepsilon \downarrow 0\end{smallmatrix}\right) \qquad\qquad \downarrow$$

$$H(T)S(T) \qquad\qquad \int_0^T H(t)\,dS(t)$$

with

$$\tau^\varepsilon_0 \triangleq 0\,; \quad \tau^\varepsilon_{2l+1} \triangleq \inf\{t > \tau^\varepsilon_{2l} : S(t) > \varepsilon\}$$

$$\tau^\varepsilon_{2l+2} \triangleq \inf\{t > \tau^\varepsilon_{2l+1} : S(t) = 0\}.$$

But how about the last term? We note

$$\int_0^T S(t)\,dH^\varepsilon(t) = \sum_{\{\ell:\tau^\varepsilon_{2\ell+1}<T\}} S(\tau^\varepsilon_{2\ell+1})\,H^\varepsilon(\tau^\varepsilon_{2\ell+1}) = \varepsilon \sum_{j=1}^{N(T,\varepsilon)} \xi_{\ell_j}$$

$$= \varepsilon N(T,\varepsilon) \cdot \frac{1}{N(T,\varepsilon)} \sum_{j=1}^{N(T,\varepsilon)} \xi_{\ell_j} .$$

Here $\{\xi_{\ell_j}\}_{j=1}^{N(T,\varepsilon)}$ is an enumeration of the $H(\tau^\varepsilon_{2\ell+1})$;

and

$$N(T,\varepsilon) \triangleq \#\{\ell : \tau^\varepsilon_{2\ell+1} < T\}$$

is the number of upcrossings of the interval $[0,\varepsilon]$, completed by S during $[0,T]$. Now the upcrossings characterization of semimartingale local time, along with the strong law of large numbers, give

$$\lim_{\varepsilon\downarrow 0} \varepsilon N(T,\varepsilon) = L^S(T) \quad \text{(in probability)}, \qquad \frac{1}{N(T,\varepsilon)} \sum_{j=1}^{N(T,\varepsilon)} \xi_{\ell_j} \xrightarrow[\varepsilon\downarrow 0]{} \mathbb{E}(\xi_1) \quad \text{(a.e.)}$$

and the claim follows. $\square$

When $\mathcal{U}$ is Brownian motion, thus S reflecting Brownian motion, the process X just constructed is what ITÔ/McKEAN (1963, 1965) called <u>SKEW BROWNIAN MOTION</u> with "skewness parameter α"; it is an ordinary Brownian motion for $\alpha = 1/2$.

With

$$B = \int_0^{\cdot} \overline{\mathrm{sgn}}\,(X(t))\, d\mathcal{U}(t)$$

then also a Brownian motion, the process X satisfies the HARRISON/SHEPP (1981) equation

$$X = B + \frac{2\alpha - 1}{\alpha} L^X .$$

This is the first instance of a stochastic equation to have been written down and studied, in which the local time of the "unknown process" X made an appearance. The authors showed that the equation admits a pathwise unique, strong solution, in fact

$$\mathcal{F}^X(t) = \mathcal{F}^B(t) \supsetneq \mathcal{F}^{\mathcal{U}}(t)$$

for $0 \le t < \infty$; showed that this solution is a diffusion; computed the scale function and speed measure; and checked that these characteristics coincided with those "posited" by ITÔ & McKEAN for their skew Brownian motion.

This work was very influential. It was followed, and very soon, by a deep study of SDE's involving the local time(s) of the unknown process (LE GALL, ENGELBERT/SCHMIDT, PERKINS, ...).

Now, one way to think of this random motion we just christened SKEW BROWNIAN, is to imagine a particle moving on the real line. As long as its distance $|X(t)|$ from the origin is positive, the particle diffuses in a "Brownian" manner: $|X(t)| = S^1(t) =$ position at time t of reflecting Brownian motion.

But when at the origin, the particle "tosses a coin" to decide whether to continue its journey in the positive half-line (probability α) or in the negative (probab. $1-\alpha$).

The origin becomes a semi-permeable boundary.

WALSH BROWNIAN MOTION

In 1978, John B. WALSH published an important paper on the skew Brownian motion. He computed not only its scale function and speed measure, but also its transition probabilities. Most remarkably, given that this was three years before the HARRISON / SHEPP paper, he considered this process not just as a diffusion but also as a semimartingale — and showed that its semimartingale local time

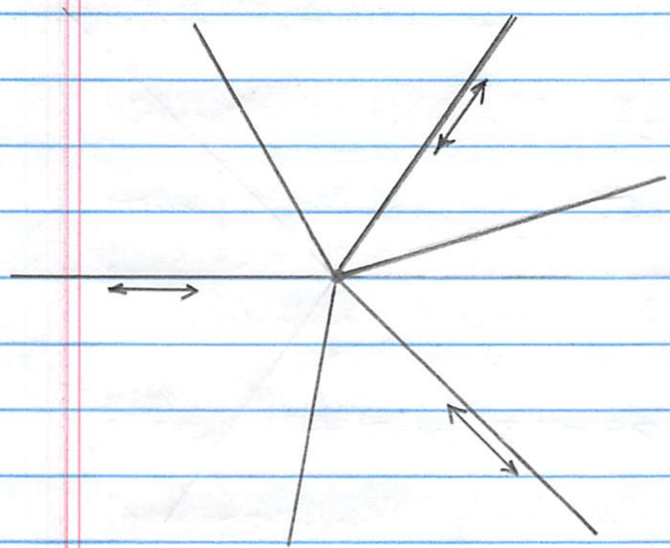
$$(t,x) \longmapsto L(t,x)$$

as in

$$\int_0^T f(X(t))\,dt = 2\int_{\mathbb{R}} f(x)\,dL(T,x),$$

the occupation-time density formula for $T \in (0,\infty)$, $f : \mathbb{R} \to [0,\infty)$, is discontinuous in the spatial variable x at the origin. The first such example; please appreciate the fact that local time was not very well understood at the time.

In the epilogue to that paper, WALSH speculated about a similar Brownian particle, now moving on the plane rather than on the real line, but in a _radial_ fashion: only along rays emanating from the origin.



Pokes of a bicycle wheel; roundhouse singularity; particle diffusing in a multipole field;

the Aeolian winds that blow your kayak in all directions at once;

the legs of a spider....

Once again, as long as away from the origin, the distance of the particle from the origin is $S(t)$: the position at time t of the reflecting Brownian motion S. When AT the origin, the particle chooses the angle for the next leg of its voyage according to a probability measure ν on $[0, 2\pi)$: independently from one visitation to the next and the rest, and independently of S.

Suppose that the measure ν charges only a finite number $\vartheta_1, \dots, \vartheta_K$ of angles, for some integer $K \geq 2$.

The resulting, so-called WALSH Brownian motion, was constructed in 1989 by BARLOW, PITMAN & YOR as a strong Markov process (with the Feller, but NOT the strong Feller, property) with the right prescription of its transition semigroup. BPY showed that the filtration generated by this WALSH-BM has the martingale representation property: Every (local) martingale can be expressed as a stochastic integral with respect to $\mathcal{U}$, the underlying scalar BM.

Now, if $K = 2$ and, say, $\vartheta_1 = 0$ and $\vartheta_2 = \pi$, we are back to the Skew-BM, whose filtration COINCIDES with that of the Brownian motion B (page 8).

Is this (stronger) property true more generally?

The answer is a very emphatic **NO.**

TSIREL'SON (GAFA '97): The filtration generated by WALSH Brownian motion with $K = 3$ (Brownian spider with three legs) cannot be generated by <u>any</u> Brownian motion, of <u>any</u> dimension.

Is there a way to construct a <u>planar unfolding</u> $X = (X_1, X_2)$ of the process

$$S = U + \max_{0 \le t \le \cdot} (-U(t)),$$ with U a given continuous semimartingale as before,

according to the spinning measure ν ?

And to do this in such a way that, when U is scalar Brownian motion, X becomes the WALSH-BM on the plane?

THEOREM (ICHIBA, K, PROKAJ, YAN 2018)

Fix a probability measure ν on $[0, 2\pi)$ and a vector $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\| = S(0)$.

Assume for concreteness, that

$$\int_0^{\infty} \mathbb{1}_{\{S(t)>0\}} \, dt = 0 \quad , \quad A \text{ is absolutely continuous with resp. to LEBESGUE}$$

$$\mathbb{P}\left(L^S(\infty) > 0\right) > 0.$$

There exists then an enlargement of the original probab. space, with a measure-preserving transformation, and on it a planar semimartingale $X = (X_1, X_2)$ with

- $\|X\| = S$ (X is a planar unfolding of S)
- The paths of X are continuous in the topology generated by the tree-metric

$$\rho(x_1, x_2) = (r_1 + r_2)\, \mathbb{1}_{\theta_1 \neq \theta_2} + |r_1 - r_2|\, \mathbb{1}_{\theta_1 = \theta_2}$$

for $x_i = (r_i, \theta_i)$, $i = 1, 2$.

(Away from the origin, X moves only along rays emanating from it.)

- For every $A \in \mathcal{B}([0, 2\pi))$, the "thinned" process

$$R^A(t) = \|X(t)\| \cdot \mathbb{1}_{\{\arg(X(t)) \in A\}}$$

is a continuous semimartingale with

$$L^{R^A}(\cdot) = \nu(A)\, L^{\|X\|}(\cdot)$$

(Thinning of Local Time)

- $$X_i = x_i + \int_0^{\cdot} \mathbb{1}_{\{X(t) \neq 0\}}\, f_i(X(t))\, dU(t) + \gamma_i\, L^{\|X\|}$$

for $i = 1, 2$. Here (Dynamics)

$f_1(x) \equiv f_1(r, \theta) \triangleq r\cos(\theta)$, $f_2(x) \equiv f_2(r, \theta) \triangleq r\sin(\theta)$

for $r > 0$.

$$\gamma_1 = \int_0^{2\pi} \cos(\theta)\, \nu(d\theta), \quad \gamma_2 \triangleq \int_0^{2\pi} \sin(\theta)\, \nu(d\theta)$$

These last "dynamics" are of the HARRISON and SHEPP type in the new, planar context.

- In fact, we have $\quad L^{X_i} = \alpha_i^{(+)} L^{S}$, $i=1,2$

with

$$\alpha_1^{(\pm)} \triangleq \int_0^{2\pi} (\cos(\theta))^{\pm} \nu(d\theta), \qquad \alpha_2^{(\pm)} \triangleq \int_0^{2\pi} (\sin(\theta))^{\pm} \nu(d\theta)$$

thus $\gamma_i = \alpha_i^{(+)} - \alpha_i^{(-)}$, and with $\alpha_i^{(+)} > 0$ we have

$$X_i = x_i + \int_0^{\cdot} \mathbb{1}_{\{X(t) \neq 0\}} f_i(X(t))\, d\mathcal{U}(t) + \left(1 - \frac{\alpha_i^{(-)}}{\alpha_i^{(+)}}\right) L^{X_i}$$

- a <u>true</u> analogue of HARRISON–SHEPP.

<u>PROPOSITION:</u> Suppose we are given a continuous, planar semimartingale $\mathcal{U}$; a vector $x = (x_1, x_2) \in \mathbb{R}^2$ with $\|x\| = S(0)$; and a vector $(\gamma_1, \gamma_2) \in \mathbb{R}^2$.

Then there exists a continuous, planar semimartingale $X = (X_1, X_2)$ that satisfies

$$\|X\| = S \quad = \text{the Skorohod reflection of } \mathcal{U}$$

as well as

$$X_i = x_i + \int_0^{\cdot} \mathbb{1}_{\{X(t) \neq 0\}} f_i(X(t))\, d\mathcal{U}(t) + \gamma_i L^{\|X\|}, \quad i=1,2$$

if, and only iff,

$$\gamma_1^2 + \gamma_2^2 \leq 1.$$

<u>PROPOSITION</u>: When $\mathcal{U}$ is scalar Brownian motion, the process X of the Theorem on pp. 14–15 is WALSH – B.M.

The dynamics of the Theorem, allow us to develop also a stochastic calculus for the so-called WALSH semimartingale $X = (X_1, X_2)$.

Consider a measurable function $g: \mathbb{R}^2 \to \mathbb{R}$ with $g(x) = g(r, \theta) = g_\theta(r)$ such that

(i) for each $\theta \in [0, 2\pi)$ the mapping $(0, \infty) \ni r \mapsto g_\theta(r)$ is the difference of two convex functions, so the derivatives $r \mapsto D^{\pm} g_\theta(r)$ exist and are of finite variation on compact intervals, and

$$D^2 g_\theta([r_1, r_2)) \triangleq D^- g_\theta(r_2) - D^- g_\theta(r_1) \quad , \quad 0 < r_1 < r_2$$

is dominated by a finite measure over some interval $[0, \eta]$; and

(ii) $\theta \mapsto D^+ g_\theta(0)$ is well-defined, bounded.

We have then the following generalized ITÔ – FREIDLIN – SHEU formula:

$$g(X(T)) = g(X(0)) + \int_0^T \mathbb{1}_{\{X(t)\neq 0\}}\, D^- g_{\Theta(t)}(\|X(t)\|)\cdot \overbrace{d\|X(t)\|}^{d\mathcal{U}(t)}$$

$$+ \left(\int_0^{2\pi} D^+ g_\theta(0)\,\nu(d\theta)\right)\cdot L^{\|X\|}(T)$$

$$+ \underbrace{\sum_{\theta\in[0,2\pi)}}_{\text{(A countable summation, over excursion intervals)}} \int_0^T \int_0^\infty \mathbb{1}_{\{X(t)\neq 0,\ \Theta(t)=\theta\}}\, \underbrace{D^2 g_\theta(dr)}_{\text{(spatial integration)}}\ \underbrace{L^{\|X\|}(dt,r)}_{\text{(temporal integration)}}$$

$\Theta(t) \triangleq \arg(X(t))$

This change of variable formula provides the basis for a stochastic calculus appropriate for such WALSH semimartingales, and the development of WALSH diffusions.

As well as the treatment of optimal stopping and control problems for such processes.

A PROBLEM OF STOCHASTIC CONTROL WITH DISCRETIONARY STOPPING

State-space: The closed unit disc $\overline{B}$, with

$$B = \{(r,\theta): 0 \le r < 1,\ 0 \le \theta < 2\pi\}.$$

On this state-space we shall consider controlled WALSH semimartingales. These will be driven by ITÔ processes

$$U = \int_0^{\cdot} \beta(t)\,dt + \int_0^{\cdot} \sigma(t)\,dW(t)$$

with local drift $\beta(\cdot)$ and local dispersion $\sigma(\cdot)$ that satisfy

$$\big(\beta(t), \sigma(t)\big) \in \mathcal{K}(\xi)\Big|_{\xi = X(t)}$$

$$\int_0^t \mathbb{1}_{\{X(u) \neq 0\}} \big(|\beta(t)| + \sigma^2(t)\big)\,dt < \infty\ , \quad \forall\ t \ge 0$$

$$X_i = x_i + \int_0^{\cdot} \mathbb{1}_{\{X(t) \neq 0\}} \Big(\beta(t)\,dt + \sigma(t)\,dW(t)\Big) + \gamma_i L^{\|X\|}\ , \quad i = 1,2$$

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with W standard, scalar Brownian motion.

Here, for every $\xi \in B \setminus \{0\}$, the set $\mathcal{K}(\xi)$ is a nonempty subset of $\mathbb{R} \times (0, \infty)$: the space of controls "available" at site ξ. NO CONTROL IS APPLIED AT THE ORIGIN; whereas we set

$$\mathcal{K}(\xi) = \{(0,0)\} \quad \text{for } \|\xi\| = 1 :$$

when X reaches the unit circumference, it gets "zapped" there (absorption).

This is the DUBINS - SAVAGE formulation of stochastic control.

In their terminology, given some $x = (x_1, x_2) \in B$, we denote by

$$\mathcal{A}(x)$$

the collection of all planar processes X with state-space $\overline{B}$ that can be expressed in the above form (on some appropriate probability space, able to support all the required processes).

There are the state-processes that are "admissible" at the initial position $X(0) = x \in B$.

For every $X \in \mathcal{A}(x)$, let us denote by $\mathcal{S}^X$ the collection of $\mathcal{F}^X(\cdot)$ - stopping times τ.

Fix now a "reward function"

$$\mathcal{U} : \bar{B} \to \mathbb{R}$$

(BOREL - measurable, bounded, continuous in the tree topology).

The value function of our stochastic control problem with discretionary stopping, is

$$V(x) = \sup_{\substack{X \in \mathcal{A}(x) \\ \tau \in \mathcal{S}^X}} \mathbb{E}\left[\mathcal{U}(X(\tau))\right].$$

We adopt here the usual convention

$$\mathcal{U}(X(\infty)) = \limsup_{t \to \infty} \mathcal{U}(X(t)).$$

Is the above supremum attained?

If so, by what controlled process $X^* \in \mathcal{A}(x)$?

what stopping rule τ^*?

Intuitively, we can write the following Dynamic Programming Variational Inequality for this problem:

$$\min\left\{ -\sup_{(\beta,\sigma)\in K(x)} \left(\beta\, D V_\theta(r) + \frac{\sigma^2}{2} D^2 V_\theta(r) \right),\ V(x) - \mathcal{U}(x) \right\} = 0$$

for $x = (r,\theta) \in B$, and

$$\min\left\{ -\int_0^{2\pi} D^+ V_\theta(0)\, \nu(d\theta),\ V(0) - \mathcal{U}(0) \right\} = 0$$

In view of the generalized ITÔ–FREIDLIN–SHEU rule of page 18, these will guarantee that

$V(X(\cdot \wedge \tau))$ will be a supermartingale, for every $X \in \mathcal{A}(x)$, $\tau \in \mathcal{S}^X$;

and that, <u>for some appropriate</u> $X^* \in \mathcal{A}(x)$,

and with

$$\tau^* = \inf\{t \geq 0 : V(X^*(t)) = \mathcal{U}(X^*(t))\}$$

the process $V(X^*(\cdot \wedge \tau^*))$ is a martingale.

- PRINCIPLE of OPTIMALITY.

Now, for $V(x) > U(x)$, this suggests

$$\sup_{(\beta,\sigma)\in \mathcal{K}(x)} \left(\frac{\beta}{\sigma^2} \cdot DV_\theta(r) + \frac{1}{2} D^2 V_\theta(r) \right) = 0$$

and thus, when $DV_\theta(r) > 0$, maximizing β/σ^2

$DV_\theta(r) < 0$, minimizing β/σ^2.

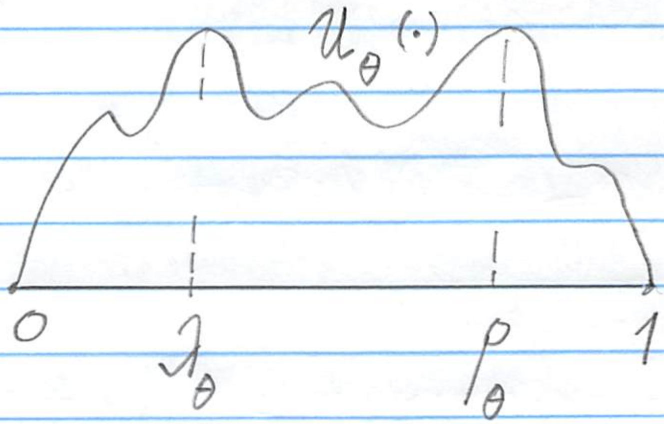
Signal-to-Noise Ratio

It turns out that a probabilistic approach can be developed, to buttress this intuition into a rigorous result; obviating the need to interpret the solvability of the HJB-type Variational Inequality on the previous page.

It goes like this:

For every $\theta \in [0, 2\pi)$ define

$$U^*_\theta = \max_{0\le r\le 1} U_\theta(r)$$



$$\lambda_\theta = \inf\{r : U^*_\theta = U_\theta(r)\}$$

$$\rho_\theta = \sup\{r : U^*_\theta = U_\theta(r)\}$$

<u>Principle:</u> To the left of λ_θ ($V_\theta(\cdot)$ ascending), you maximize the signal-to-noise ratio;

to the right of r_θ ($V_\theta(\cdot)$ descending), you minimize the signal-to-noise ratio;

in between, you select any strategy that will bring you to one of there points.

- Assume there exist pairs of measurable functions (b_0, s_0), (b_1, s_1) with

$$\frac{b_0(x)}{s_0^2(x)} = \inf\left\{\frac{\beta}{\sigma^2} : (\beta, \sigma) \in \mathcal{K}(x)\right\}, \quad (b_0(x), s_0(x)) \in \mathcal{K}(x)$$

$$\frac{b_1(x)}{s_1^2(x)} = \sup\left\{\frac{\beta}{\sigma^2} : (\beta, \sigma) \in \mathcal{K}(x)\right\}, \quad (b_1(x), s_1(x)) \in \mathcal{K}(x)$$

- For any real $c \geq \mathcal{U}(0)$, denote by $(b^{(c)}, s^{(c)})$ a pair of measurable functions with

$$\begin{aligned}
(b^{(c)}, s^{(c)}) &= (b_1, s_1), \quad \text{for } \mathcal{U}^*_\theta > c, \quad 0 < r < \lambda_\theta \\
&= (b_0, s_0), \quad \text{for } \mathcal{U}^\theta_* \geq c, \quad \rho_\theta < r < 1 \\
&\qquad\qquad\qquad \text{or } \mathcal{U}^*_\theta < c, \quad 0 < r < 1
\end{aligned}$$

And for this pair $(b^{(c)}, s^{(c)})$, introduce the scale function

$$p^{(c)}(\theta, r) = p^{(c)}_\theta (r) = \int_0^r \exp\left(-2 \int_0^y \frac{b^{(c)}(z,\theta)}{(s^{(c)}(z,\theta))^2}\, dz \right) dy$$

- Finally, for every real constant $c \geq \mathcal{U}(0)$, define the least $p^{(c)}$ - concave envelope of $\mathcal{U}$

$$\mathcal{U}^{(c)}(x) = \mathcal{U}^{(c)}(\theta, r) = \mathcal{U}^{(c)}_\theta (r) \triangleq \inf \{ \varphi(r) : \varphi(\cdot) \geq \mathcal{U}_\theta(\cdot),\ \varphi : [0,L] \to \mathbb{R} \text{ is } p^{(c)}_\theta\text{ - concave},\ \varphi(0) \geq c \}.$$

(We say that a function φ is p - concave, if $\varphi(\xi) = g(p(\xi))$ for some concave g.)

THEOREM: The value function V of the control problem with discretionary stopping, is given by

$$V(x) = \mathcal{U}^{(c_*)}(x), \qquad c_* \triangleq \inf \left\{ c \geq \mathcal{U}(0) : \int_0^{2\pi} D^+ \mathcal{U}^{(c)}_\theta (0)\, \nu(d\theta) \leq 0 \right\}$$

and thus $V(0) = c_*$.

The supremum on page 21 is attained by the WALSH diffusion X^* with local drift function $b^{(c_*)}(\cdot)$, local dispersion function $s^{(c_*)}(\cdot)$, and spinning measure ν; and by the stop-rule

$$\tau_* = \inf\{t \geq 0 : V(X^*(t)) = u(X^*(t))\}.$$

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