Algebraic Topology QR Exam – January 2022

1. (a) State the definition of a *CW complex* and its topology (the *weak topology*).

   (b) Define the 2-sphere S^2, up to homeomorphism, to be the set

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

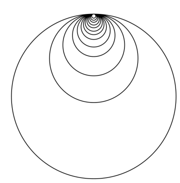
   topologized as a subspace of Euclidean 3-space. Give a rigorous proof that the 2-sphere admits a CW complex structure, verifying that its topology agrees with the weak topology on your chosen CW complex. [You may take for granted standard results from point-set topology, and standard results about continuity of maps between subspaces of Euclidean space. For the remainder of the exam you may take for granted standard results about CW complex structures on common spaces.]

2. Let G be a graph, that is, a 1-dimensional CW complex. Let S^2 denote the 2-sphere. For each of the following statements, either prove the statement, or give (with justification) a counterexample.

   (a) Every continuous map $G \to S^2$ is nullhomotopic.

   (b) Every continuous map $S^2 \to G$ is nullhomotopic.

3. For $n \in \mathbb{N}$, let C_n be the metric circle of radius $\frac{1}{n}$ in $\mathbb{R}^2$ with its north pole at the origin $(0, 0)$. Let C be the union $\bigcup_n C_n$, topologized as a subspace of Euclidean 2-space. The space C has been called the *infinite earring*, the *Hawaiian earring*, and the *shrinking wedge of circles*.



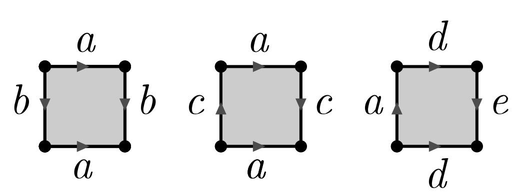
   The space C is a standard example of a space that is not *semi-locally simply connected*. Prove that C does not have a universal cover by verifying that it is not semi-locally simply connected, and proving that every space with a universal cover is semi-locally simply connected.

4. Suppose that a certain space X decomposes as the union of three open subsets, $X = U_1 \cup U_2 \cup U_3$, satisfying the following properties.

   - The open sets U_1, U_2, and U_3 are contractible.
   - The pairwise intersections $U_1 \cap U_2$, $U_1 \cap U_3$, and $U_2 \cap U_3$ are contractible.
   - The triple intersection $U_1 \cap U_2 \cap U_3$ is empty.

   Prove that X has the same homology as the circle S^1.

5. A space Y is constructed by gluing together a torus, a Klein bottle, and a cylinder along the edges labelled a below, i.e., Y is constructed from three squares using the edge identifications shown.



   (a) Calculate a presentation for the fundamental group of Y.

   (b) Calculate the homology of Y.

Solutions

1. (a) A CW complex is a (filtered) topological space X defined as follows. Its *0-skeleton* X^0 is a discrete set of points. We inductively define the n-skeleton X^n by gluing closed n-disks to the $(n-1)$-skeleton along their boundaries. Specifically, to construct X^n we first take the disjoint union of the $(n-1)$-skeleton X^{n-1} and a collection of disjoint n-disks $\{D^n_\alpha\}_\alpha$. Associated to each n-disk we define an *attaching map*, a continuous map $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$. We then define the n-skeleton X^n as the quotient space

$$X^n = \left(X^{n-1} \bigsqcup_\alpha D^n_\alpha \right) \Big/ \; x \sim \phi_\alpha(x) \text{ for all } \alpha \text{ and } x \in \partial D^n_\alpha,$$

with the quotient topology.

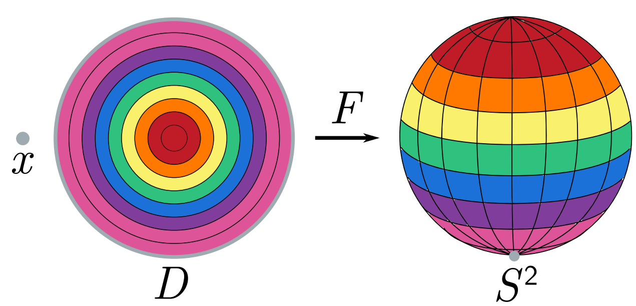
We let $X = \bigcup_{n \geq 0} X^n$. We endow X with the *weak topology*: a subset $U \subseteq X$ is open if and only if $U \cap X^n$ is open for every n. We note that, if X is finite dimensional (i.e. X equals its d-skeleton X^d for some d), then the weak topology agrees with the quotient topology defined on X^d above.

(b) There are many choices of CW complex structure on S^2; in this sample solution we will examine the CW complex with one 0-disk x and one 2-disk D. The attaching map $\phi : \partial D \to x$ is necessarily the constant map, and the CW complex $X = X^2$ is defined as the quotient of $D \sqcup x$ obtained by gluing the boundary ∂D to x,

$$\begin{aligned} X = X^2 &= (x \sqcup D) \;/\; y \sim x \text{ for all } y \in \partial D \\ &= (x \sqcup D) \;/\; (x \sqcup \partial D). \end{aligned}$$

We must prove that X is homeomorphic to S^2. First we define a map $F : x \sqcup D \to S^2$. Identify D with the unit disk in $\mathbb{R}^2$, described in polar coordinates (r, ϕ). To simplify the formula we parameterize $S^2 \subseteq \mathbb{R}^3$ using spherical coordinates (r, θ, ϕ), where $r \geq 0$ is the vector's length, $\theta \in [0, \pi]$ its angle to the positive z-axis, and $\phi \in [0, 2\pi)$ the angle from the positive x-axis to its projection in the xy-plane. The sphere S^2 is the level set $r = 1$. Define F by

$$\begin{aligned} F : x \sqcup D &\longrightarrow S^2 \\ x &\longmapsto (1, \pi, 0) \qquad \text{[the south pole of } S^2\text{]} \\ (r, \phi) &\longmapsto (1, r\pi, \phi) \end{aligned}$$



By construction, the map F surjects onto S^2. The map F is injective except at the points $\{x\} \cup \{(r, \phi) \mid r = 1\} = x \sqcup \partial D$, which all map to the south pole of S^2. Thus by the universal property of the quotient topology, the map F factors through a continuous map f from the quotient $X = (x \sqcup D) \;/\; (x \sqcup \partial D)$,

$$\begin{tikzcd} x \sqcup D \arrow[r, "q"] \arrow[dr, "F"'] & X = (x \sqcup D)/(x \sqcup \partial D) \arrow[d, "f"] \\ & S^2 \end{tikzcd}$$

By construction, the map $f : X \to S^2$ is bijective. The quotient space X is the continuous image $q(x \sqcup D)$ of the compact space $x \sqcup D$ and is therefore compact. Every continuous bijective map from a compact space to a Hausdorff space is a homeomorphism. Thus f is a homeomorphism, and we have constructed a CW complex structure on S^2 as desired.

2. (a) The statement is **true.**

   Let $f : G \to S^2$ be a continuous map. Recall that the 2-sphere S^2 admits a CW complex structure with one 0-cell and one 2-cell. The cellular approximation theorem states,

   > **Theorem.** Every continuous map $X \to Y$ of CW complexes is homotopic to a cellular map.

   Thus f is homotopic to a continuous function that maps the 1-skeleton of G (that is, all of G) to the 1-skeleton of S^2 (that is, a single point). In other words, f is nullhomoptic.

   (b) The statement is **true.**

   Let $g : S^2 \to G$ be a continuous map. Its image $g(S^2)$ must be connected, so we can just consider the connected component of G containing $g(S^2)$, and WLOG assume G is connected.

   The graph G is a connected CW complex, therefore it is path-connected, locally path-connected, and semi-locally simply connected. Thus G has a universal cover $p : \widetilde{G} \to G$. Since G is a 1-dimensional CW complex, so are its covers. Its universal cover $\widetilde{G}$ is a simply connected graph, so $\widetilde{G}$ is a tree, and therefore contractible.

   The sphere S^2 has trivial fundamental group, so $g_*(\pi_1(S^2)) = 0 \subseteq p_*(\pi_1(\widetilde{G}))$. Then by the lifting criterion for covering spaces, the map g factors through a map $\tilde{g} : S^2 \to \widetilde{G}$,

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ S^2 \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & G \end{tikzcd}$$

   Since $\widetilde{G}$ is contractible, the map $\tilde{g}$ is nullhomotopic via some homotopy h_t. But then $p \circ h_t$ is a nullhomotopy of the map g.

3. Let X be a path-connected, locally path-connected topological space. Recall that the universal cover $p : \widetilde{X} \to X$ of X is, if it exists, a covering map such that the covering space $\widetilde{X}$ is simply connected. By the lifting criterion and the unique lifting property for covering spaces, the universal cover (if it exists) is unique up to isomorphism.

   Consider the following definition.

   > **Definition.** Let X be a topological space. Then X is *semi-locally simply connected* if every point $x \in X$ has a neighbourhood U with the property that every loop in U is nullhomotopic in X. Equivalently, the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ has trivial image.

   We first prove that the space C is not semi-locally simply connected. Let $x = (0, 0)$ be the origin. Fix a neighbourhood U of x in C. The neighbourhood U must contain the circle C_N for some N sufficiently large. Consider the continuous retraction map $\phi : C \to C_N$ that is the identity on C_N and maps the circle C_n to the point x for all $n \neq N$. Then the composition

$$C_N \hookrightarrow U \hookrightarrow C \xrightarrow{\phi} C_N$$

is the identity map on C_N. Therefore the composition

$$\mathbb{Z} \cong \pi_1(C_N, x) \to \pi_1(U, x) \to \pi_1(C, x) \xrightarrow{\phi_*} \pi_1(C_N, x) \cong \mathbb{Z}$$

is the identity map, and we deduce that the map $\pi_1(U, x) \to \pi_1(C, x)$ is nonzero, as desired.

Next we will argue that a space X with a universal cover must be semi-locally simply connected. Suppose $p : \widetilde{X} \to X$ is the universal cover. Choose a point $x \in X$, and choose a preimage $\widetilde{x} \in p^{-1}(x)$. By definition of a covering space, there exists a neighbourhood U of x and lift $\widetilde{U}$ such that $\widetilde{U}$ is a neighbourhood of $\widetilde{x}$ and $p|_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism.

Let $\gamma : S^1 \to U$ be any loop in U; we will show γ is nullhomotopic in X. The map $(p|_{\widetilde{U}})^{-1} \circ \gamma$ is a lift of γ to $\widetilde{U}$. Since $\widetilde{X}$ is simply connected by assumption, there is a nullhomotopy F_t from $(p|_{\widetilde{U}})^{-1} \circ \gamma$ to the constant map at some point $\widetilde{c} \in \widetilde{X}$. But then $p \circ F_t$ is a homotopy in X from $p \circ (p|_{\widetilde{U}})^{-1} \circ \gamma = \gamma$ to the constant map at $p(\widetilde{c}) \in X$. Thus γ is nullhomotopic in X and it follows that X is semi-locally simply connected.

We conclude that the space C does not have a universal cover.

4. We will prove this result using three applications of the Mayer–Vietoris long exact sequence.

   We have a choice between using the version of the Mayer–Vietoris sequence for reduced or un-reduced homology. In this solution we will work with reduced homology, noting that (by definition of reduced homology) the empty space has reduced homology groups

$$\widetilde{H}_i(\varnothing) = \begin{cases} \mathbb{Z}, & i = -1 \\ 0, & i \neq -1. \end{cases}$$

   Since the reduced homology groups determine the un-reduced homology groups up to isomorphism (by adding a direct factor of $\mathbb{Z}$ in degree 0), to solve the problem it suffices show that

$$\widetilde{H}_i(X) \cong \widetilde{H}_i(S^1) \cong \begin{cases} \mathbb{Z}, & i = 1 \\ 0, & i \neq 1. \end{cases}$$

   Recall that, given open subsets A and B in a topological space, the Mayer–Vietoris sequence is the long exact sequence on homology groups

$$\cdots \longrightarrow \widetilde{H}_n(A \cap B) \longrightarrow \widetilde{H}_n(A) \oplus H_n(B) \longrightarrow \widetilde{H}_n(A \cup B) \xrightarrow{\delta} \widetilde{H}_{n-1}(A \cap B) \longrightarrow \cdots$$

   Claim 1. $\widetilde{H}_i(U_1 \cup U_2) = 0$ for all i.

   We first consider the Mayer–Vietoris long exact sequence in the case

$$A = U_1 \simeq * \quad \text{and} \quad B = U_2 \simeq *.$$

   By hypothesis, $A \cap B = U_1 \cap U_2 \simeq *$. Thus, for all n, we have an exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow & \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) & \longrightarrow & \widetilde{H}_n(A \cup B) & \xrightarrow{\delta} & \widetilde{H}_{n-1}(A \cap B) & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ & 0 & & \widetilde{H}_n(U_1 \cup U_2) & & 0 & \end{array}$$

   and we deduce that $U_1 \cup U_2$ is acyclic; $\widetilde{H}_n(U_1 \cup U_2) = 0$ for all n as claimed.

   Claim 2. $\widetilde{H}_i((U_1 \cup U_2) \cap U_3) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i \neq 0. \end{cases}$

We next consider the Mayer–Vietoris long exact sequence in the case $A = U_1 \cap U_3 \simeq *$ and $B = U_2 \cap U_3 \simeq *$. Then

$$A \cap B = U_1 \cap U_2 \cap U_3 = \varnothing, \qquad A \cup B = (U_1 \cap U_3) \cup (U_2 \cap U_3) = (U_1 \cup U_2) \cap U_3$$

Thus

$$\begin{aligned} &H_i(A) = 0 \text{ for all } i \\ &H_i(B) = 0 \text{ for all } i \\ &H_i(A \cap B) = 0 \text{ for all } i \neq -1 \\ &H_{-1}(A \cap B) = \mathbb{Z} \end{aligned}$$

and the Mayer–Vietoris sequence is, for $n \geq 1$,

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \arrow[r] \arrow[d, equal] & \widetilde{H}_n(A \cup B) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{H}_{n-1}(A \cap B) \arrow[r] \arrow[d, equal] & \cdots \\ & 0 & \widetilde{H}_n((U_1 \cup U_2) \cap U_3) & 0 & \end{tikzcd}$$

and for $n = 0$,

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \arrow[r] \arrow[d, equal] & \widetilde{H}_0(A \cup B) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{H}_{-1}(A \cap B) \arrow[r] \arrow[d, equal] & \widetilde{H}_{-1}(A) \oplus \widetilde{H}_{-1}(B) \arrow[r] \arrow[d, equal] & \cdots \\ & 0 & \widetilde{H}_0((U_1 \cup U_2) \cap U_3) & \mathbb{Z} & 0 & \end{tikzcd}$$

We deduce $\widetilde{H}_i((U_1 \cup U_2) \cap U_3) \cong \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & i \neq 0. \end{cases}$

Claim 3. $\widetilde{H}_i(X) = \widetilde{H}_i(U_1 \cup U_2 \cup U_3) \cong \begin{cases} \mathbb{Z}, & i = 1 \\ 0, & i \neq 1. \end{cases}$

Finally we apply the Mayer–Vietoris sequence to the subsets $A = U_1 \cup U_2$ and $B = U_3$. The space B is acyclic by assumption, and A is acyclic by Claim 1. We computed the reduced homology of the intersection $A \cap B = (U_1 \cup U_2) \cap U_3$ in Claim 2.

For $n \geq 2$, we have

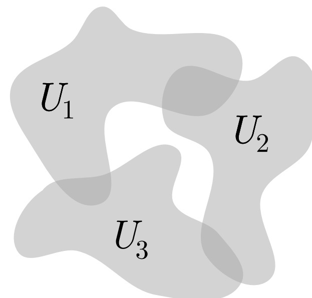
$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \arrow[r] \arrow[d, equal] & \widetilde{H}_n(A \cup B) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{H}_{n-1}(A \cap B) \arrow[r] \arrow[d, equal] & \cdots \\ & 0 & \widetilde{H}_n(X) & 0 & \end{tikzcd}$$

Thus $\widetilde{H}_n(X)$ for all $n \geq 2$. For $n = 1, 0$ we have

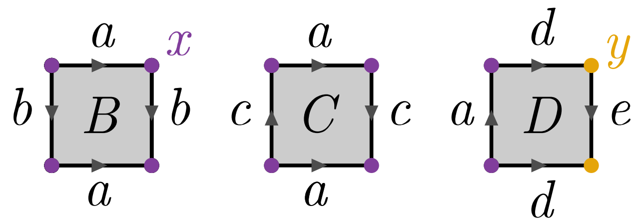
$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \arrow[r] \arrow[d, equal] & \widetilde{H}_1(A \cup B) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{H}_0(A \cap B) \arrow[r] \arrow[d, equal] & \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \arrow[d, equal] \\ & 0 & \widetilde{H}_1(X) & \mathbb{Z} & 0 \end{tikzcd}$$

$$\begin{tikzcd} \arrow[r] & \widetilde{H}_0(A \cup B) \arrow[r, "\delta"] \arrow[d, equal] & \widetilde{H}_{-1}(A \cap B) \arrow[r] \arrow[d, equal] & \cdots \\ & \widetilde{H}_0(X) & 0 & \end{tikzcd}$$

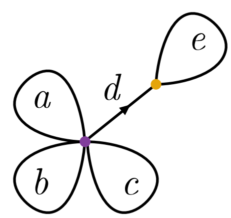
Thus $\widetilde{H}_1(X) \cong \mathbb{Z}$ and $\widetilde{H}_0(X) = 0$. This concludes Claim 3 and the proof.



5. When we trace out the orbits of the corner points under the edge identifications, we see that there are two vertices (labelled x and y below). Thus Y is a CW complex with two 0-cells x, y, five 1-cells a, b, c, d, e, and three 2-cells which we label B, C, D below.



(a) The 1-skeleton Y^1 of Y is shown below.



The edge d is the unique choice of maximal tree. By abuse of notation, we also write a (respectively, b, c, etc) to denote the element of $\pi_1(Y, x)$ represented by the inclusion of the (directed) edge a. Then $\pi_1(Y^1, x)$ is freely generated by the four loops a, b, c, e', where $e' = ded^{-1}$, and it follows that the loops a, b, c, e' generate $\pi_1(Y, x)$.

Each 2-cell determines to a relator in $\pi_1(Y, x)$. Each 2-disk is glued down by its boundary along a word in the letters a, b, c, d, e; we must re-express these words (up to equivalence as elements in $\pi_1(Y, x)$) as words in the generators $a, b, c, e' = ded^{-1}$. The cell B is glued along the word $aba^{-1}b^{-1}$. The cell C is glued along the word $aca^{-1}c$. The cell D is glued along the word $ded^{-1}a = e'a$. Thus

$$\pi_1(Y) \cong \langle\, a, b, c, e' \mid aba^{-1}b^{-1},\ aca^{-1}c,\ e'a \,\rangle.$$

Since the last relation states that $e' = a^{-1}$, we can (optionally) simplify this presentation,

$$\pi_1(Y) \cong \langle\, a, b, c \mid aba^{-1}b^{-1},\ aca^{-1}c \,\rangle.$$

(b) Again view Y as a CW complex with two 0-cells x, y, five 1-cells a, b, c, d, e, and three 2-cells B, C, D. Its cellular chain complex is:

$$0 \longrightarrow C_2(Y) \xrightarrow{\quad\partial_2\quad} C_1(Y) \xrightarrow{\quad\partial_1\quad} C_0(Y) \longrightarrow 0$$

$$
\begin{aligned}
A &\longmapsto a + b - a - b = 0 \\
B &\longmapsto a + c - a + c = 2c \\
C &\longmapsto d + e - d + a = e + a
\end{aligned}
$$

$$
\begin{aligned}
a &\longmapsto x - x = 0 \\
b &\longmapsto x - x = 0 \\
c &\longmapsto x - x = 0 \\
d &\longmapsto y - x \\
e &\longmapsto y - y = 0
\end{aligned}
$$

To compute the kernel of ∂_2, suppose for some $m, n, k \in \mathbb{Z}$,

$$
\begin{aligned}
\partial_2(mA + nB + kC) &= 0 \\
m\,\partial_2(A) + n\,\partial_2(B) + k\,\partial_2(C) &= 0 \\
m0 + n(2c) + k(e + a) &= 0 \\
2nc + ke + ka &= 0
\end{aligned}
$$

Since c, e, a are linearly independent in $C_1(Y)$ this implies $n = k = 0$. Thus $\ker(\partial_2) = \langle A \rangle$. Similarly the kernel of ∂_1 is $\langle a, b, c, e \rangle$. Thus,

$$H_0(Y) = \frac{C_0(Y)}{\mathrm{im}(\partial_1)} = \frac{\mathbb{Z}\{x, y\}}{\langle x - y \rangle} \cong \mathbb{Z}$$

$$H_1(Y) = \frac{\ker(\partial_1)}{\mathrm{im}(\partial_2)} = \frac{\mathbb{Z}\{a, b, c, e\}}{\langle 2c, e + a \rangle} \cong \frac{\mathbb{Z}\{a, b, c\}}{\langle 2c \rangle} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$$

$$H_2(Y) = \ker(\partial_2) = \langle A \rangle \cong \mathbb{Z}$$

We can verify the calculation of $H_0(Y)$ and $H_1(Y)$ using part (a). By its construction Y is a connected CW complex, hence path-connected, so necessarily $H_0(Y) \cong \mathbb{Z}$. We can calculate $H_1(Y)$ by abelianizing the presentation for $\pi_1(Y)$,

$$\pi_1(Y) \cong \langle\, a, b, c \mid aba^{-1}b^{-1},\ aca^{-1}c \,\rangle.$$

If the generators a, b, c commute, then the relator $aba^{-1}b^{-1}$ vanishes and the relator $aca^{-1}c$ simplifies to c^2. Thus (switching from multiplicative to additive notation) again we see

$$H_1(Y) \cong \frac{\mathbb{Z}\{a, b, c\}}{\langle 2c \rangle} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}.$$