General and Differential Topology QR Exam – May 2, 2022

This exam consists of five problems. All manifolds are assumed to be C^∞. $\mathfrak{X}(M)$ denotes the space of smooth vector fields on the manifold M. All items will be graded independently of each other.

Problem 1. The subgroup Γ of $\mathrm{SU}(2)$ generated by the matrix

$$\gamma := \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SU}(2)$$

is isomorphic to $\mathbb{Z}_4$, and it acts on the unit three-sphere $S^3 \subset \mathbb{C}^2$ by matrix multiplication. Let $X = S^3/\Gamma$ be the orbit space, with the quotient topology. Answer the following questions, with proofs:

1. Is X is second countable?
2. Is X is Hausdorff?
3. Is the projection $\pi : S^3 \to X$ is a local homeomorphism?

Problem 2. A smooth $F : M \to M$ is called a *Lefshetz map* iff for all $p \in M$ such that $F(p) = p$ one has:

$$1 \in \mathbb{R} \text{ is not an eigenvalue of } \quad dF_p : T_pM \to T_pM.$$

1. Show that F is Lefschetz iff its graph and the diagonal $\Delta \subset M \times M$ intersect transversely.
2. Show that if F is Lefschetz then the set $\{p \in M : F(p) = p)\}$ consists of isolated points.
3. Is the converse of the previous statement true?

Problem 3. Let $S^2 \subset \mathbb{R}^3$ be the two-sphere with the standard orientation, and $F : S^2 \to S^2$ be given by $F(a, b, c) = (a, -b, -c)$. Note that F is a diffeomorphism (no proof needed). Also, let x, y, z denote the restrictions of the coordinate functions to S^2, and let $\alpha = xydy \wedge dz$.

1. Establish whether or not F is orientation preserving, and compute $F^*(\alpha)$. What does your findings imply about the value of $\int_{S^2} \alpha$? Explain.
2. The vector field $\langle -y, x, 0 \rangle$ in $\mathbb{R}^3$ is tangent to the sphere, and therefore it restricts to a vector field $X \in \mathfrak{X}(S^2)$. Compute the Lie derivative $\mathcal{L}_X \alpha$.

–over please–

Problem 4. Let M be a smooth manifold, and $X \in \mathfrak{X}(M \times \mathbb{R})$ be a smooth vector field of the form

$$\forall (p,s) \in M \times \mathbb{R} \qquad X_{(p,s)} = (V_{(p,s)}, \partial_s), \qquad \text{where} \quad V_{p,s} \in T_pM.$$

(We are identifying $T_{(p,s)}(M \times \mathbb{R})$ with $T_pM \times T_s\mathbb{R}$.) For each $(p,s) \in M \times \mathbb{R}$, let $t \mapsto \Phi_t(p,s)$ be the integral curve of X starting at (p,s), and denote

$$\phi_{t,s}(p) := \pi\left(\Phi_{t-s}(p,s)\right), \quad \text{where} \quad \pi : M \times \mathbb{R} \to M \quad \text{is the projection.}$$

1. Show that $\forall t_0 \in \mathbb{R}$, $p \in M$ the curve on M $t \mapsto \gamma(t) = \phi_{t,t_0}(p)$ is defined in a neighborhood of t_0 and satisfies $\dot{\gamma}(t) = V_{\gamma(t),t}$, $\gamma(t_0) = p$.

2. Assuming that X is complete, show that $\forall r, s, t \in \mathbb{R}$

   $$\phi_{t,s} \circ \phi_{s,r} = \phi_{t,r},$$

   where $\phi_{t,s} : M \to M$ is the map $p \to \phi_{t,s}(p)$, etc.

Problem 5. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$ (where e is the identity), and let $A, B \in \mathfrak{g}$ be linearly independent elements satisfying $[A, B] = 0$.

1. Carefully explain why $\forall s, t \in \mathbb{R}$ $\exp(sA)\exp(tB) = \exp(tB)\exp(sA)$, where $\exp : \mathfrak{g} \to G$ is the exponential map.

2. Show that the map $E : \mathbb{R}^2 \ni (s,t) \mapsto \exp(sA)\exp(tB) \in G$ is a an immersion.

3. Specialize to the case $G = \mathrm{U}(3)$, and A, B diagonal with diagonal entries the components of $i\vec{\lambda} = \langle i\lambda_1, i\lambda_2, i\lambda_3\rangle$ and $i\vec{\mu} = \langle i\mu_1, i\mu_2, i\mu_3\rangle$ respectively.

   Under what conditions on $\vec{\lambda}, \vec{\mu} \in \mathbb{R}^3$ is the image of the map E a closed (regular) submanifold of $\mathrm{U}(3)$?