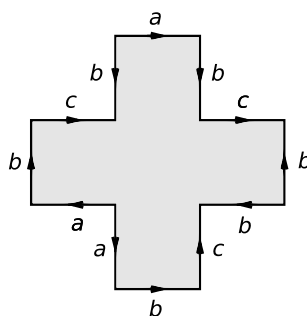# Algebraic Topology QR Exam – May 2022

1. Let X be the quotient space defined by the following polygon with edge identifications. Compute $\pi_1(X)$.


2. Let X be the wedge of 2 circles a and b with single vertex v. By mild abuse of notation we write a and b to mean both the edges of the graph X and the corresponding generators of $\pi_1(X, v)$. Consider the following two covers $p : \tilde{X} \to X$, with the map p specified by the edge labels and orientations of $\tilde{X}$. A distinguished lift $\tilde{v}$ of v is marked with a gray dot. Let $H = p_*(\pi_1(\tilde{X}, \tilde{v}))$. For each cover, state with (very) brief justification,

   (i) a free generating set for H,

   (ii) the index of H as a subgroup of $\pi_1(X, v)$,

   (iii) whether the cover is regular,

   (iv) the deck group of the cover (as an abstract group),

   (v) generators for the normalizer of H in $\pi_1(X, v)$.
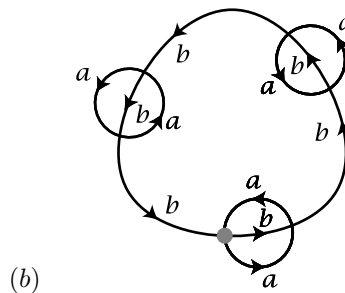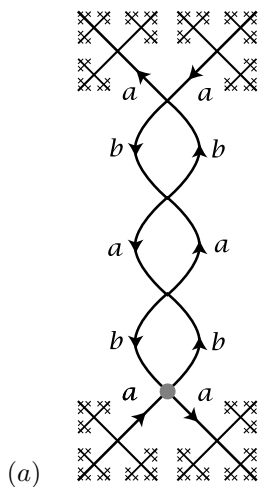
3. Let $p : \tilde{X} \to X$ be a covering map of path-connected spaces, and let $\tilde{x}_0 \in \tilde{X}$. Let $x_0 = p(\tilde{x}_0)$. For each of the following statements: either prove the statement using the definition and/or lifting properties of a covering space, or construct a counterexample. In a proof, give complete theorem statements for any lifting properties you cite.

   (i) The induced map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

   (ii) The induced map $p_* : H_1(\tilde{X}) \to H_1(X)$ is injective.

4. Let N be a positive integer. Let X be the 2-sphere, and let $A \subseteq X$ be the union of N disjoint circles of latitude (pictured below for $N = 3$). Let X/A be the quotient space with A collapsed to a point. Compute $\tilde{H}_*(X/A)$.
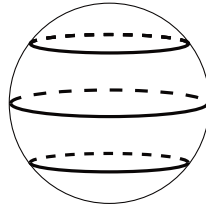
5. Let M be an n-manifold for some $n \geq 1$, and let $x \in M$. Consider the pair $(X, A) = (M, M \backslash \{x\})$.

   (a) Consider the quotient space X/A where A is collapsed to a point. Describe the topology on X/A and show (by writing an explicit homotopy and verifying continuity) that this space is contractible.

   (b) Prove that $H_*(X, A)$ and $\tilde{H}_*(X/A)$ are not equal.