SEPTEMBER 2016

SOLUTIONS

1. Let $M(n,\mathbb{R})$ be the space of all real $n \times n$ matrices and $GL(n,\mathbb{R})$ be the subset of invertible matrices. Let $X \in GL(n,\mathbb{R})$ and $B \in M(n,\mathbb{R})$ Show that

$$\frac{d}{dt}|_{t=0} \det(X \cdot e^{tB}) = \det X \operatorname{trace}(B).$$

Show that $SL(n,\mathbb{R})$ is a closed submanifold of $GL(n,\mathbb{R})$ of dimension $n^2 - 1$.

Solution: Since $\frac{d}{dt}|_{t=0} \det(X \cdot e^{tB}) = \det(X) \frac{d}{dt}|_{t=0} \det(e^{tB})$, it suffices to show

$$\frac{d}{dt}|_{t=0} \det(e^{tB}) = \operatorname{trace}(B).$$

For a diagonal matrix D with eigenvalues λ_i, the matrix e^{tD} is diagonal with eigenvalues $e^{t\lambda_i}$. So

$$\frac{d}{dt}|_{t=0} \det(e^{tD}) = \frac{d}{dt}|_{t=0} e^{t\lambda_1 + \cdots + t\lambda_n} = \lambda_1 + \ldots \lambda_n = \operatorname{trace}(D).$$

If B is diagonalizable, write $B = P^{-1}DP$ for $P \in GL(n,\mathbb{R})$ and D a diagonal matrix. This gives

$$e^{tP^{-1}DP} = \sum_{n=0}^{\infty} \frac{(tP^{-1}DP)^n}{n!} = \sum_{n=0}^{\infty} \frac{P^{-1}(tD)^n P}{n!} = P^{-1}e^{tD}P.$$

We then have

$$\frac{d}{dt}|_{t=0} \det(e^{tB}) = \frac{d}{dt}|_{t=0} \det(e^{tD}) = \operatorname{trace}(D) = \operatorname{trace}(B).$$

Hence the desired equality holds for diagonalizable matrices. Since diagonalizable matrices are dense and the functions in the equality are continuous, the equality must hold for all matrices.

Now we will show $SL(n,\mathbb{R})$ is a closed submanifold of $GL(n,\mathbb{R})$. Let $f : GL(n,\mathbb{R}) \to \mathbb{R}$ be given by $f(A) = \det(A)$. Then $SL(n,\mathbb{R}) = f^{-1}(1)$, which is a closed subset of $GL(n,\mathbb{R})$. To see it is a submanifold, we must verify that 1 is a regular value of f. Let $X \in SL(n,\mathbb{R})$ and let $B \in T_X GL(n,\mathbb{R}) \cong M(n,\mathbb{R})$. Then

$$D_X F(B) = \frac{d}{dt}|_{t=0} F(Xe^{tB}) = \det(X)\operatorname{trace}(B) = \operatorname{trace}(B).$$

Thus $D_X F$ is surjective onto $\mathbb{R}$. Indeed let B be a matrix with top left entry $\lambda \in \mathbb{R}$ and zeros everywhere else. This shows 1 is a regular value of f and hence by the regular value theorem, $f^{-1}(1)$ is a smooth submanifold of $GL(n,\mathbb{R})$ of codimension 1.

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and let C be the cubic surface defined by

$$C = \{y^2 x = x^3 - xz^2\}.$$

Define $X = S^2 \cap C$. Is X a smooth manifold of $\mathbb{R}^3$?

Solution: Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f = (f_1, f_2)$, where

$$f_1(x, y, z) = x^2 + y^2 + z^2 - 1$$
$$f_2(x, y, z) = xy^2 - x^3 + xz^2.$$

Then $S^2 \cap C = f^{-1}(0, 0)$.
Note that $f_2(x, y, z) = x(y^2 - x^2 + z^2)$. Substituting the defining equation $1 - x^2 = y^2 + z^2$ for S^2, we get

$$0 = x(1 - 2x^2).$$

This gives $x = 0, \pm\frac{\sqrt{2}}{2}$. We have $f_1(0, y, z) = y^2 + z^2 - 1$, $f_2(0, y, z) = 0$, and $f_1(\pm\frac{\sqrt{2}}{2}, 0, 0) = y^2 + z^2 - 1/2$, $f_2(\pm\frac{\sqrt{2}}{2}, 0, 0) = \pm\frac{\sqrt{2}}{2}(y^2 + z^2 - 1/2)$. So $S^2 \cap C$ is the union of three circles:

$$x = \pm\frac{\sqrt{2}}{2},\ y^2 + z^2 = 1/2$$

and

$$x = 0,\ y^2 + z^2 = 1,$$

which means $S^2 \cap C$ is a smooth submanifold of $\mathbb{R}^3$.

3. Consider $\mathbb{R}^{2n}$ with coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and define the 1-form α by

$$\alpha = \sum_i y_i dx_i,$$

and the 2-form ω by

$$\omega = d\alpha.$$

(1) Let V_x be the subspace $\{y = 0\} \subset \mathbb{R}^{2n}$ and $\iota_x : V_x \to \mathbb{R}^{2n}$ the inclusion, and similarly for V_y, ι_y. Show the pullbacks $\iota_x^*\omega$ on V_x and $\iota_y^*\omega$ on V_y are identically 0.

Solution: By definition of the pullback, $\iota_x^*\omega(v, w) = \omega(d\iota_x(v), d\iota_x(w))$ and similarly for y. Let $v, w \in T_pV_x$. Then the vectors $d\iota_x(v), d\iota_x(w)$ have y-coordinates all 0, i.e. $dy_i(d\iota_x(v)) = dy_i(d\iota_x(w)) = 0$ for all i. Note

$$\omega = \sum_i dy_i \wedge dx_i,$$

so $\omega(d\iota_x(v), d\iota_x(w)) = 0$ for any $v, w \in T_pV_x$. The same argument works for T_pV_y with the roles of x and y reversed.

(2) Let $S^* = \{(x, y) \mid y_1^2 + \cdots + y_n^2 = 1\}$. Show that the $2n - 1$ form

$$\alpha \wedge (\omega)^{n-1} = \alpha \wedge \omega \wedge \cdots \wedge \omega \ (n-1)\,\text{times}$$

is nowhere zero on the submanifold S^*.

Solution: Since $\omega = \sum_i dy_i \wedge dx_i$, we compute

$$(\omega)^{n-1} = c(n) \sum_i (-1)^{i-1} dy_1 \wedge dx_1 \wedge \ldots \widehat{dy_i \wedge dx_i} \cdots \wedge dy_n \wedge dx_n,$$

where $c(n)$ is a constant depending on n, and $\widehat{dy_i \wedge dx_i}$ means $dy_i \wedge dx_i$ is deleted from the expression. Hence

$$\alpha \wedge (\omega)^{n-1} = c(n) \sum_i y_i \, dy_1 \wedge dx_1 \wedge \dots \widehat{dy_i} \wedge dx_i \cdots \wedge dy_n \wedge dx_n.$$

Let $\beta_i = dy_1 \wedge dx_1 \wedge \dots \widehat{dy_i} \wedge dx_i \cdots \wedge dy_n \wedge dx_n$. Let $v_1, \dots, v_{2n-1}$ be tangent vectors to S^* and let A be the $2n \times (2n-1)$ matrix whose columns are the v_i expressed in terms of the coordinates $(y_1, x_1, \dots, y_n, x_n)$ on $\mathbb{R}^{2n}$. Then $\beta_i(v_1, \dots, v_{2n-1})$ is the determinant of the $(2n-1) \times (2n-1)$ matrix obtained from A by deleting the row corresponding to the y_i coordinate. By the cofactor expansion formula for the determinant, it follows that $\sum_i y_i \beta_i(v_1, \dots, v_{2n-1})$ is the determinant of the $2n \times 2n$ matrix obtained by taking A and adding the column $v = (y_1, 0, \dots, y_n, 0)$ at the end. If $v_1, \dots, v_{2n-1}$ form a basis for the tangent space of S^* at the point $(y_1, 0, \dots, y_n, 0)$, then $v_1 \dots, v_{2n-1}, v$ forms a basis for $\mathbb{R}^{2n}$, hence the determinant will be nonzero. This shows $\alpha \wedge (\omega)^{n-1}$ is nonzero on S^*.

(3) Write down a vector field ξ tangent to S^* which is not identically 0 so that for every vector field η tangent to S^* we have $\omega(\xi, \eta) \equiv 0$.

Solution: Write $\xi = \sum_i a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$ and $\eta = \sum_i c_i \frac{\partial}{\partial x_i} + d_i \frac{\partial}{\partial y_i}$. The tangency to S^* condition means $\sum_i y_i b_i = \sum y_i d_i = 0$. We have

$$\omega(\xi, \eta) = \sum_i b_i c_i - a_i d_i.$$

Letting $a_i = y_i$ and $b_i = 0$ for all i gives $\omega(\xi, \eta) \equiv 0$ for any c_i, d_i.

4. Let M be a smooth manifold, $A \subset M$ a closed subset and $U \supset A$ an open neighborhood of A in M. Suppose that f is a smooth real-valued function defined on U. Show that there is a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f} \equiv f$ on a neighborhood of A.

Solution: The open sets U and $M \setminus A$ cover M. So there exists a C^∞ partition of unity subordinate to this open covering, i.e. smooth functions ρ_1, ρ_2 on M so that

(1) $0 \leq \rho_i \leq 1$
(2) $\mathrm{supp}\rho_1 \subset U$, $\mathrm{supp}\rho_2 \subset M \setminus A$
(3) $\rho_1(x) + \rho_2(x) = 1$ for all $x \in M$

Let $\tilde{f} = \rho_1 \cdot f$. This is well-defined since ρ_1 is 0 outside of U. By condition 2, we know ρ_2 is 0 on a neighborhood of A. Since $\rho_1 + \rho_2 \equiv 1$, we must have $\rho_1 \equiv 1$ on a neighborhood A. So $\tilde{f} \equiv f$ on a neighborhood of A.

5. Let $O(3) \subset GL(3, \mathbb{R})$ be the 3×3 orthogonal group. Let $\omega = g^{-1} dg$ be the 3×3 matrix of one-forms on $O(3)$, where

$$g = \begin{pmatrix} g_{1,1} & \dots & g_{1,3} \\ & & \\ g_{3,1} & \dots & g_{3,3} \end{pmatrix}, dg = \begin{pmatrix} dg_{1,1} & \dots & dg_{1,3} \\ & & \\ dg_{3,1} & \dots & dg_{3,3} \end{pmatrix},$$

and the $g_{i,j}$ are coordinate functions in $M(3,\mathbb{R})$. Finally for $a \in O(3)$ fixed, let $L_a : O(3) \to O(3)$ be given by left multiplication, i.e. $L_a(h) = a \cdot h$ for all $h \in O(3)$. Show that $L_a^*\omega = \omega$, i.e. ω is left-invariant.

Solution: First we find ω_{ij}, the ijth entry of the matrix ω:

$$\omega_{ij} = \sum_{k=1}^{3} (g^{-1})_{i,k}(dg)_{k,j} = \sum_{k=1}^{3} g_{k,i} dg_{k,j},$$

where the last equality uses $g^{-1} = g^T$, since the matrix g is orthogonal. Then

$$\begin{aligned} L_a^*\omega_{ij} &= \sum_{k=1}^{3} (g_{k,i} \circ L_a)(dg_{k,j} \circ dL_a) \\ &= \sum_{k=1}^{3} (g_{k,i} \circ L_a) d(g_{k,j} \circ L_a). \end{aligned}$$

The matrix with ijth entry equal to $g_{i,j} \circ L_a$ is the same as the matrix ag. This means the ijth entry of $L_a^*\omega$ corresponds to the ijth entry of the matrix $(ag)^T d(ag) = g^{-1}a^{-1}a\, dg = g^{-1}\, dg$ as desired.