THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

September 3, 2016: Morning Session, 9:00 to 12:00 noon.

1. Let $Z = \{(x, y) \in \mathbb{C}^2 \mid x = 0 \text{ or } y = 0\}$. Find the homology of $\mathbb{C}^2 \setminus Z$ (with the subspace topology induced from the Euclidean topology on $\mathbb{C}^2$).

2. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Consider in the torus $T = S^1 \times S^1$ the homeomorph of the open disk
$$U = \{(x, y) \mid Re(x), Re(y) > 1/2\}.$$
Consider a homeomorphism $h : S^1 \to \partial U$ where ∂U is the boundary of the closure of U in T, and let $f : S^1 \to \partial U$ be the map given by $f(z) = h(z^2)$. Now let X be the quotient of
$$D \amalg (T \setminus U)$$
by the smallest equivalence relation which has $z \sim f(z)$ for $z \in S^1 \subset D$. Find $\pi_1(X)$ in terms of generators and relations.

3. Give an example of a (connected) 3-fold covering which is not regular.

4. Let $Y = (S^1 \times S^1)/(S^1 \times \{1\})$ (*i.e.*, collapse $S^1 \times \{1\}$ to a point) with the quotient topology. Find the homology of Y.

5. For given $k, n \geq 1$, construct a topological space M such that $\widetilde{H}_k(M) = \mathbb{Z}/n$ and $\widetilde{H}_i(M) = 0$ for all $i \neq k$.

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Qualifying Review examination in Topology

September 3, 2016: Afternoon Session, 2:00 to 5:00.

1. Let $M(n,\mathbb{R})$ be the space of all real $n\times n$ matrices, and $Gl(n,\mathbb{R})\subset M(n,\mathbb{R})$ be the subset of invertible matrices. Let $X\in Gl(n,\mathbb{R})$ and $B\in M(n,\mathbb{R})$. Show that

$$\frac{d}{dt}\det(X\cdot e^{tB})|_{t=0}=\det X\ \mathrm{Trace}(B).$$

Show that $Sl(n,\mathbb{R}):=\{A\in M(n,\mathbb{R})|\det A=1\}$ is a closed submanifold of $Gl(n,\mathbb{R})$, of dimension n^2-1.

2. Let $S^2\subset\mathbb{R}^3$ be the unit sphere, and let C be the cubic surface defined by

$$C=\{(x,y,z)\in\mathbb{R}^3\,|\,y^2x=x^3-xz^2\}$$

Define $X=S^2\cap C$. Is X a smooth submanifold of $\mathbb{R}^3$?

3. Consider $\mathbb{R}^{2n}$ with coordinates $(x,y)=(x_1,\dots,x_n,y_1,\dots,y_n)$ and define the 1-form α by

$$\alpha=\sum_i y_i\,dx_i,$$

and the 2-form ω by

$$\omega=d\alpha.$$

Let V_x be the subspace $\{y=0\}\subset\mathbb{R}^{2n}$ and $\iota_x:V_x\to\mathbb{R}^{2n}$ the inclusion, and similarly for V_y,ι_y. Show that the pull-backs $\iota_x^*\omega$ on V_x and $\iota_y^*\omega$ on V_y are identically zero. Let $S^*=\{(x,y)\,|\,y_1^2+\cdots+y_n^2=1\}$. Show that the $2n-1$-form

$$\alpha\wedge(\omega)^{n-1}=\alpha\wedge\omega\wedge\cdots\wedge\omega\ \ (n-1\text{ times})$$

is nowhere zero on the submanifold S^*. Write down a vector field ξ tangent to S^* which is not identically 0 such that for every vector field η tangent to S^*, we have $\omega(\xi,\eta)\equiv 0$.

4. Let M be a smooth manifold, $A\subset M$ a closed subset and $U\supset A$ an open neighborhood of A in M. Suppose that f is a smooth, real-valued function defined on U. Show that there is a smooth function $\tilde{f}:M\to\mathbb{R}$ such that $\tilde{f}\equiv f$ on a neighborhood of A.

5. Let $O(3) \subset Gl(3,\mathbb{R})$ be the 3×3 orthogonal group. Let $\omega = g^{-1} \cdot dg$ be the 3×3 matrix of one-forms on $O(3)$, where

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,3} \\ & & \\ g_{3,1} & \cdots & g_{3,3} \end{pmatrix}, \text{ and } dg = \begin{pmatrix} dg_{1,1} & \cdots & dg_{1,3} \\ & & \\ dg_{3,1} & \cdots & dg_{3,3} \end{pmatrix},$$

and the $g_{i,j}$ are the coordinate functions in $M(3,\mathbb{R})$. Finally, for $a \in O(3)$ fixed, let $L_a : O(3) \to O(3)$ be given by left multiplication, i.e.,

$$L_a(g) = a \cdot g.$$

Show that $L_a^* \omega = \omega$, *i.e.*, ω is left invariant.