UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Applied Analysis
6 May 2005: Morning Session, 9:00-12:00

1. Consider the heat equation $u_t = u_{xx}$ for $0 < x < 1$ and $t > 0$, supplemented with the boundary conditions

$$u(0,t) + 2u_x(0,t) = 0 \quad \text{and} \quad u(1,t) + 2u_x(1,t) = 0\,,$$

and the initial condition

$$u(x,0) = f(x)\,.$$

The boundary condition at $x = 1$ is of the form of Newton's law of cooling; the rate that heat escapes from the rod at this end is proportional to the temperature at the end of the rod. On the other hand, the boundary condition at $x = 0$ is "backwards"; the rate that heat *enters* the rod at this end is proportional to the temperature at the end of the rod.

(a) Find the solution for general $f(x) \in L^2(0,1)$.

(b) If $f(x)$ is a piecewise smooth function having jump discontinuities at several points in the interval $(0,1)$, how many of the derivatives $\partial^n u/\partial x^n$ will be continuous functions of x for $t > 0$? Why?

(c) What is the solution in the special case when $f(x) \equiv 1$? Give full details. Describe the solution asymptotically as $t \to \infty$.

(d) Consider the heat equation with the modified boundary conditions

$$u(0,t) + 2u_x(0,t) = 1 \quad \text{and} \quad u(1,t) + 2u_x(1,t) = 1\,,$$

but with zero initial temperature: $u(x,0) = 0$. Show how the solution of this problem is related to the solution of part (c).

2. Consider the Laplacian $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ acting on functions of $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$. Define

$$G(\mathbf{x}) := \frac{e^{-\kappa^2 r}}{r}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ with the parameter $\kappa > 0$.

(a) Prove, in the sense of distributions, that $(-\nabla^2 + \kappa^2)G(\mathbf{x}) = 4\pi\delta(\mathbf{x})$. You may use the fact that in spherical coordinates (using the standard notation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$),

$$\nabla^2 f(\mathbf{x}) = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f\,.$$

(b) Find a solution of the inhomogeneous equation $(-\nabla^2 + \kappa^2)\Phi = \rho$ where $\rho = \rho(r)$ is a function of r alone with suitable decay for large r. Show that if $\rho(r) \equiv 0$ for $r > R$, then there is a solution Φ that for $r > R$ is of the form $\Phi(\mathbf{x}) = MG(\mathbf{x})$ for some constant M. Give a formula for M in terms of the function $\rho(r)$.

3. (a) Let $\{f_n\}$ be a sequence in $L^2(a,b)$. Suppose that $f_n \to f$ in $L^2(a,b)$ as $n \to \infty$, and show that for every $g \in L^2(a,b)$, $\langle f_n, g\rangle \to \langle f, g\rangle$ as $n \to \infty$.

(b) Show that for all $f, g \in L^2(a,b)$,

$$\Big|\,\|f\| - \|g\|\,\Big| \le \|f - g\|\,.$$

(c) Prove that if $f_n \to f$ in $L^2(a,b)$, then $\|f_n\| \to \|f\|$. (That is, the $L^2(a,b)$ norm is a continuous functional with respect to convergence in $L^2(a,b)$.)

4. In the following $V(x)$ is a smooth function, periodic on $\mathbb{R}$ with period one, restricted to a fundamental period $x \in [0,1]$. Consider the initial-boundary-value problem $u_t = V''(x)u + V'(x)u_x + u_{xx}$ with periodic boundary conditions on $[0,1]$ and smooth initial data $u(x,0) = u_0(x)$.

   (a) Show that
   $$\int_0^1 u(x,t)dx = \int_0^1 u_0(x)\,dx$$
   for all $t > 0$.

   (b) Show that $Ce^{-V(x)}$ is a stationary solution of the partial differential equation for any constant C.

   (c) Suppose the solution $u(x,t)$ with positive initial data $u_0(x) > 0$ is smooth for all $t > 0$. Show then that $u(x,t) > 0$ for all $t > 0$. Hint: consider $u_t(x,t)$ at points x where u might vanish.

   (d) Consider the functional
   $$S[f] = \int_0^1 f(x) \ln\left[e^{V(x)} f(x)\right] dx$$
   defined for positive functions $f(x)$. Show that
   $$\frac{d}{dt} S[u(\cdot,t)] \le 0$$
   and use this, together with the results above, to prove that starting with positive initial data $u_0(x)$ there exists a long-time limit for $u(x,t)$. Compute this limit, and show how it depends on $u_0(x)$.

5. (a) For each $\epsilon > 0$, the function
   $$f_\epsilon(x) := \frac{1}{2\epsilon} e^{-|x|/\epsilon}$$
   defines a distribution in $\mathcal{D}'(\mathbb{R})$ (recall $\mathcal{D}(\mathbb{R})$ is the space of test functions in $C^\infty(\mathbb{R})$ with compact support) by the usual rule
   $$F_\epsilon[\phi] := \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)\,dx$$
   assigning a numerical value to each test function $\phi(x) \in \mathcal{D}(\mathbb{R})$. Prove that
   $$\lim_{\epsilon \to 0} F_\epsilon = \delta$$
   in $\mathcal{D}'(\mathbb{R})$, where δ denotes the delta distribution defined on $\mathcal{D}(\mathbb{R})$ by $\delta[\phi] = \phi(0)$.

   (b) The function $f(x)$ defined by
   $$f(x) := \begin{cases} 0\,, & x \le 0 \\ x^{-1/2}\,, & x > 0\,, \end{cases}$$
   is in $L^1_{\text{loc}}(\mathbb{R})$ and thus defines a distribution F by the usual rule
   $$F[\phi] := \int_{-\infty}^{\infty} f(x)\phi(x)\,dx = \int_0^\infty \frac{\phi(x)}{x^{1/2}}\,dx\,.$$
   Students of calculus will tell you that $f(x)$ is differentiable for $x \neq 0$, and that
   $$f'(x) = \begin{cases} 0\,, & x < 0 \\ -\frac{1}{2}x^{-3/2}\,, & x > 0\,. \end{cases}$$
   Unlike $f(x)$, this function blows up fast enough as $x \to 0$ that $f'(x)$ is not in $L^1_{\text{loc}}(\mathbb{R})$. Thus, $f'(x)$ cannot define a distribution in the same way that $f(x)$ does. Find the distributional derivative $(F')[\phi]$ and show how it relates to the calculus derivative $f'(x)$.

(c) The distribution $F_n := e^{2n}\delta_n$, where δ_n denotes the delta distribution centered at $x = n$, may be considered either as a distribution from the space $\mathcal{D}'(\mathbb{R})$, or as a *tempered distribution* from the space $\mathcal{S}'(\mathbb{R})$ dual to the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions. Prove that $F_n \to 0$ as $n \to \infty$ in the sense of convergence in $\mathcal{D}'(\mathbb{R})$. Then show (perhaps by finding an appropriate Schwartz function giving a counterexample) that the sequence $\{F_n\}$ has no limit in the sense of convergence in $\mathcal{S}'(\mathbb{R})$.

UNIVERSITY OF MICHIGAN
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Qualifying Review Examination in Applied Analysis
6 May 2005: Afternoon Session, 2:00-5:00

1. Consider the initial-value problem

$$\frac{dy}{dt} + y = 1, \qquad y(0) = 0,$$

which we intend to solve numerically using the difference scheme

$$y_0 = 0, \qquad \frac{y_1 - y_0}{h} + y_1 = 1, \qquad \frac{y_{n+1} - y_{n-1}}{2h} + y_n = 1, \qquad n \geq 1.$$

(a) Determine the order of accuracy of the scheme.

(b) If you program and run this scheme on a computer, you will find that the numerical solution eventually develops a sawtooth oscillation (*i.e.* a high-frequency, $(-1)^n$ oscillation). Explain why this happens. Also explain why the exact solution does not have a sawtooth oscillation.

(c) If time step h is reduced, the oscillations occur at a later time. Explain why this happens.

(d) Suggest an alternative (consistent) scheme which will not produce any oscillations, independent of the time step.

2. Consider the explicit scheme

$$r_j^{n+1} = \frac{1}{2}\left(r_{j+1}^n + r_{j-1}^n\right) + \frac{\Delta t}{2\Delta x}\left(s_{j+1}^n - s_{j-1}^n\right), \quad \text{and} \quad s_j^{n+1} = \frac{1}{2}\left(s_{j+1}^n + s_{j-1}^n\right) + \frac{\Delta t}{2\Delta x}\left(r_{j+1}^n - r_{j-1}^n\right),$$

where periodic boundary conditions are imposed on the sequences $\{r_j^n\}$ and $\{s_j^n\}$ regarding their dependence on j for each fixed n.

(a) With which system of partial differential equations is this scheme consistent?

(b) Find conditions on Δt and Δx sufficient for this scheme to satisfy the von Neumann stability condition. Describe how this calculation relates to L^2-stability. Also, find the modified equations for the scheme and show how your stability analysis relates to them.

3. Consider solving the heat equation $u_t = u_{xx}$ with boundary conditions $u(0,t) = u(1,t) = 0$ and the numerical scheme

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x^2}\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right), \qquad u_0^n = u_N^n = 0.$$

(a) This is an implicit scheme. Hence, to advance the solution from time $t = n\Delta t$ to $t = (n+1)\Delta t$, consider the iteration

$$u_j^{n+1,k+1} = u_j^n + \frac{\Delta t}{\Delta x^2}\left(u_{j-1}^{n+1,k} - 2u_j^{n+1,k} + u_{j+1}^{n+1,k}\right),$$

for $k = 0, 1, 2, 3, \ldots$, and with $u_j^{n+1,0}$ arbitrary. The idea is that by repeating this iteration, one should have $u_j^{n+1} = u_j^{n+1,\infty}$. Under what conditions on Δt and Δx does convergence occur as $k \to \infty$? Explain.

(b) What is the order of the local truncation error for this scheme?

4. Consider the heat equation $u_t = u_{xx}$ approximated by Euler's method in time t and second-order centered differences in space x.

   (a) Use the energy method to prove that the scheme is conditionally stable in the ℓ^2-norm.

   (b) Let $u(x,0) = \sin x$, and discretize this initial condition by setting $u_j^0 = u(jh, 0)$. Find explicitly both the exact solution $u(x,t)$ for $t > 0$ and the numerical solution u_j^n for $n > 0$. Prove that

$$\lim_{h \to 0} u_j^n = u(x,t)\,,$$

   where $x = jh$, $t = nk > 0$, and $\lambda = k/h^2 > 0$ are all held fixed.

   (c) Part (b) shows that the scheme converges for any positive value of λ, including values for which the scheme is unstable. Does this contradict the Lax equivalence theorem? Explain.

5. Consider the initial-value problem

$$y'' + \omega^2 y = 0\,, \qquad y(0) = y_0\,, \qquad y'(0) = {y_0}'\,.$$

   (a) Introducing $v = y'$, write this as a first order system, and obtain the formulas for solving the problem numerically using Euler's method with step size h.

   (b) Find the region of absolute stability for the numerical method from part (a).

   (c) Consider instead the implicit numerical method

$$\begin{pmatrix} y_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} y_n + h v_n \\ v_n - h\omega^2 y_{n+1} \end{pmatrix}.$$

   What is the order of accuracy of this method?

   (d) Find the region of absolute stability of the difference scheme in part (c).