Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 5, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that for each $z_0 \in \mathbb{C}$ the power series expansion

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has at least one coefficient a_n which is zero. Show that f is a polynomial.

Problem 2: Let U be an open subset of $\mathbb{C}$ and $z_0 \in U$. Suppose that $f(\cdot)$ is a meromorphic function on U with a pole at z_0. Prove there is no holomorphic function $g : U - \{z_0\} \to \mathbb{C}$ such that $f(z) = e^{g(z)},\ z \in U - \{z_0\}$.

Problem 3: Use contour integration to evaluate the integral

$$\int_0^{\infty} \frac{x^{-1/3}}{1+x}\, dx \ .$$

Problem 4: Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ denote the unit disc. Consider a sequence of holomorphic functions $f_n : \mathbb{D} \to \mathbb{C}$ having the Taylor expansions

$$f_n(z) = \sum_{k=1}^{\infty} a_{n,k} z^k$$

where $|a_{n,k}| \le k^{2022}$ for all $k, n \in \mathbb{N}$.
Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ contains a subsequence uniformly convergent on compact subsets of $\mathbb{D}$.

Problem 5: Find the number of solutions (counted according to multiplicity) in the annulus $\{1 \le |z| \le 3\}$ of the equation $z^9 + z^6 + 30z^5 - 3z + 2 = 0$.