Complex Analysis Qualifying Review

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1. Let D be the ellipse $\frac{x^2}{1} + \frac{y^2}{4} < \frac{1}{2021}$ in the complex plane (so $z = x + iy$), and let $z_1, z_2 \in D$ be two distinct points. Let $\varphi \colon D \to D$ be an analytic (=holomorphic) map such that $\varphi(z_1) = z_1$ and $\varphi(z_2) = z_2$. Prove that φ is the identity map.

2. Let $D = \mathbb{D}(0, r)$ be a disc centered at the origin, and let f be a function defined and analytic in a neighborhood of the closure of D. Assume that $|f(z)| < r^2$ when $|z| = r$. Prove that there exists $\varepsilon > 0$ such that if $|\zeta| \leq \varepsilon$, then the equation $f(z) = z^2 + \zeta$ has exactly two solutions (with multiplicity) in D.

3. Let a and b be complex numbers with $0 < |a| < |b|$. Find three different Laurent series expansions of the rational function $f(z) = \frac{1}{(z-a)(z-b)}$, valid in three different regions, each of which is invariant under rotation around the origin.

4. Let $\mathbb{H} = \{\operatorname{Re} z > 0\}$ be the right half plane, and f an analytic function on $\mathbb{H}$. Assume that $f(z) \leq \frac{1}{(\operatorname{Re} z)^2}$ for all $z \in \mathbb{H}$. Prove that $|f'(1)| \leq \frac{27}{4}$.

5. Let $D \subset \mathbb{C}$ be a domain (i.e. a connected open set) and $(g_n)_n$ a sequence of uniformly bounded analytic functions on D. Assume that there exists a point $\zeta \in D$ such that for all $m \geq 0$, the derivatives $g_n^{(m)}(\zeta)$ converge to zero as $n \to 0$. Prove that $(g_n)_n$ converges locally uniformly on D (i.e. uniformly on each compact subset of D) to 0.