Complex Analysis Qualifying Review

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1. Let $\alpha > 0$ be a real number.

(a) Prove that if $\alpha \leq 1$, then there exists an analytic function f on the unit disc such that $f(\frac{1}{n}) = \frac{1}{n+\alpha}$ for all integers $n \geq 1$.
(b) Prove that if $\alpha > 1$ and f is an analytic function on the unit disc, then there exist only finitely many integers $n \geq 1$ such that $f(\frac{1}{n}) = \frac{1}{n+\alpha}$.

2. Does there exist an entire function f (i.e. f is analytic in the whole complex plane) such that the inequality

$$\tfrac{1}{2}|z|^{3/2} - |z| \leq |f(z)| \leq 2|z|^{3/2} + \tfrac{7}{2}|z|$$

holds for all z outside a compact set? Justify your answer.

3. Find all analytic functions f on the unit disc $\mathbb{D}$ such that $f(0) = 1$, $f(\frac{1}{2}) = 3$, and $\mathrm{Re} f(z) > 0$ for all $z \in \mathbb{D}$.

4. Use complex integration to compute the real integral $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

5. Let D be the (open) square with corners at $\pm 1 \pm i$. Find the number of solutions to the equation $e^z = 3z^{2020}$ in D, counted with multiplicity.