Analysis Qualifying Review

Saturday, September 2, 2017
Morning Session, 9:00 AM - Noon

1. Let S denote the strip $\{z = x + iy\colon 0 < y < 1\} \subset \mathbb{C}$. Construct an explicit analytic bijection $f\colon S \to S$ extending continuously to $\overline{S} \setminus \{0\}$ but not to all of $\overline{S} =$ the closure of S in $\mathbb{C}$. (Compositions of explicit maps are allowed.)

2. Suppose that we have nowhere-vanishing entire functions g_j and polynomials p_j of degree ten so that the entire functions $f_j \stackrel{\text{def}}{=} g_j \cdot p_j$ converge uniformly on each compact subset of $\mathbb{C}$ to a limit function f. Show that f may be written in the form $f = g \cdot p$ with g a nowhere-vanishing entire function and p a polynomial. Could we have $\deg p > 10$? Could we have $\deg p < 10$?

3. Let A denote the annulus $\{z \in \mathbb{C}\colon 1 < |z| < 3\}$. Exhibit a vector space V of analytic functions on A with the property that an analytic function f on A is the second derivative of some analytic function on A if and only if

$$\int_{|z|=2} f(\zeta) h(\zeta)\, d\zeta = 0$$

for all $h \in V$.

4. Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected open set, and let $f : \Omega \to \Omega$ be an analytic mapping. Show that if f has two distinct fixed points, then f is the identity mapping.

5. Compute the definite integral

$$I = \lim_{R \to +\infty} \int_0^R \frac{x \sin x}{1 + x^2}\, dx.$$

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Saturday, September 2, 2017
Afternoon Session, 2:00 - 5:00 PM

N.B.: "Measure" means "Lebesgue measure" throughout. Similarly, $L^p(\mathbb{R}^n)$ is always with respect to Lebesgue measure on $\mathbb{R}^n$.

1. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of measurable functions. Prove that the set

$$E = \{x \in [a,b] \mid \lim f_n(x) \text{ exists } \in \mathbb{R} \cup \{\pm\infty\}\}$$

is measurable.

2. Let ϕ: $[-1,1] \to [-1,1]$ be a non-decreasing function with $\phi(-1) = -1$ and $\phi(1) = 1$. Show that

$$\int_{(x,y)\in[-1,1]^2} (\phi(x) - \phi(y))^2 \, dA(x,y) \leq 8.$$

When will equality hold?

3. Construct a function $f \in L^1(\mathbb{R}^2)$ such that $f \notin L^2(D(x_0, r))$ for any disk $D(x_0, r) \subseteq \mathbb{R}^2$, where x_0 is the center of the disk and $r > 0$ is its radius.

4. Let $f \in L^2(I)$, for any finite interval $I \subset \mathbb{R}$. Assume that

$$\int_{-a}^{a} |t||f(x+t)| \, dt \geq \frac{2}{\sqrt{3}} a^2,$$

for all $a > 0, x \in \mathbb{R}$. Show that $|f(x)| \geq 1$ for a.e. $x \in \mathbb{R}$.

5. Calculate the Fourier transform

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) \, e^{i\,x\,\xi} \, dx,$$

for

$$f(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \text{ for fixed } t > 0.$$

You may use the standard integral

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$