Algebra I Exam: May 2022

Please wait until directed to begin the exam. Please use the indicated page, and its reverse side, for your solution to each problem. Extra pages are attached at the end if you need more space; please indicate if you have used them.

Please write your **identification number** here:

Have fun!

Problem 1. Let A be an $n \times n$ integer matrix and let A^T be its transpose. Let X and Y be the abelian groups $X = \mathbb{Z}^n / A\mathbb{Z}^n$ and $Y = \mathbb{Z}^n / A^T\mathbb{Z}^n$. Show that X and Y are isomorphic as abelian groups.

Problem 2. Let k be a field. For each of the following rings, determine if it is a PID or a UFD or neither or both.

(1) $k[x, y]$.

(2) $k[x, y]/(xy - 1)k[x, y]$.

(3) $k[x, y]/(y^2 - x^3)k[x, y]$.

Problem 3. Let T be an $(n \times n)$-matrix over an algebraically closed field k of characteristic p. Assume that all eigenvalues of T lie in $\mathbb{F}_p \subset k$. Is the matrix $T^p - T$ nilpotent? If yes, give a proof; if not, give an example.

Problem 4. Calculate the number of subgroups $L \subset \mathbb{Z}^3$ with $\mathbb{Z}^3/L$ being isomorphic abstractly to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 5. Let R be a PID which is free as rank n as a $\mathbb{Z}$-module and let π be a prime element of R. Show that $|R/\pi R|$ is of the form p^k for some prime integer p and some $1 \leq k \leq n$. (Remark: Note that units, and the zero element, are not considered prime.)