# Algebra I Exam – May 2021

Notation: $\mathbb{C}$ and $\mathbb{Q}$ denote the fields of complex and rational numbers.

**Problem 1.** Let $I$ be the ideal of $\mathbb{C}[x, y, z]$ generated by the elements

$$x + 2y - z, \quad 2x + y + z, \quad (x + y + 3z)(1 + 2x - y + 2z).$$

Find all maximal ideals that contain $I$.

**Problem 2.** Let $V$ be a non-zero complex vector space, let $n$ be a positive integer, let $\alpha \in \bigwedge^n V$, and let $v$ be a non-zero vector in $V$. Show that $\alpha \wedge v = 0$ if and only if $\alpha = \beta \wedge v$ for some $\beta \in \bigwedge^{n-1} V$.

**Problem 3.** Let $R$ be a commutative ring containing the field $\mathbb{C}$. Suppose that

$$0 \to N \to E \to M \to 0$$

is a short exact sequence of $R$-modules such that $N$ and $M$ are non-isomorphic and one-dimensional over $\mathbb{C}$. Show that the sequence splits (as a sequence of $R$-modules).

**Problem 4.** Let $M$ be an abelian group with a subgroup $N$ such that $M/N \cong \mathbb{Q}$. Show that the natural map $N/kN \to M/kM$ is an isomorphism for any positive integer $k$.

**Problem 5.** Let $R$ be a commutative ring and let $f_1, f_2, \ldots$ be an infinite sequence of elements in $R$. Suppose that for each $N \geq 1$ there exists a field $K_N$ and a unital ring homomorphism $\phi_N \colon R \to K_N$ such that $\phi_N(f_1) = \cdots = \phi_N(f_N) = 0$. ("Unital" just means that $\Phi_N(1) = 1$.) Show that there exists a field $K$ and a ring homomorphism $\phi \colon R \to K$ such that $\phi(f_i) = 0$ for all $i \geq 1$.