August 2020, Qualifying Review Algebra, Part I

Problem 1. Let V and W be finite dimensional complex vector spaces, and let $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Let T be the tensor $v_1 \otimes w_1 + v_2 \otimes w_2$ in $V \otimes W$. Show that, if T is of the form $x \otimes y$ for some $x \in V$ and $y \in W$, then either v_1 and v_2 are linearly dependent or else w_1 and w_2 are linearly dependent.

Solution. Proceeding by contradiction, suppose v_1 and v_2 are linearly independent and w_1 and w_2 are linearly independent. We can then find bases $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_m$ of V and W. The tensors $v_i \otimes w_j$ are a basis for the tensor product $V \otimes W$. By assumption, we have an expression

$$v_1 \otimes w_1 + v_2 \otimes w_2 = \left(\sum_{i=1}^{n} a_i v_i \right) \otimes \left(\sum_{j=1}^{m} b_j w_j \right)$$

Expanding the product and equating coefficients, we find

$$a_1 b_1 = 1, \quad a_1 b_2 = 0, \quad a_2 b_1 = 0, \quad a_2 b_2 = 1.$$

This is a contradiction: the first and fourth equations imply that a_1, a_2, b_1, and b_2 are non-zero, and therefore $a_1 b_2$ cannot be zero.

Problem 2. Let I be the ideal $\langle x^2+1, y^2+1 \rangle$ in the ring $\mathbf{Q}[x,y]$. Show that I is not prime, and give a prime ideal containing I.

Solution. The ideal I contains the element $(x^2+1) - (y^2+1) = (x-y)(x+y)$. We claim that $x-y$ and $x+y$ do not belong to I. Indeed, let $f \colon \mathbf{Q}[x,y] \to \mathbf{C}$ be the unique ring homomorphism with $f(x) = i$ and $f(y) = -i$. Then $f(x^2+1) = 0$ and $f(y^2+1) = 0$, so $f(I) = 0$, but $f(x-y) = 2i \neq 0$; thus $x - y \notin I$. The proof for $x+y$ is similar. We conclude that I is not prime.

Let J be the ideal of $\mathbf{Q}[x,y]$ generated by x^2+1, y^2+1, and $x-y$. Of course, J contains I. We have $\mathbf{Q}[x,y]/J \cong \mathbf{Q}[x]/(x^2+1)$ since $x = y$ in $\mathbf{Q}[x,y]/J$. Since x^2+1 is an irreducible polynomial over $\mathbf{Q}$, the quotient $\mathbf{Q}[x]/(x^2+1)$ is a domain, and so J is prime.

Problem 3. Let $p(x,y)$ be an irreducible polynomial with complex coefficients. Let R be the subring of $\mathbf{C}(x,y)$ consisting of all rational functions $\frac{f(x,y)}{g(x,y)}$ such that $p(x,y) \nmid g(x,y)$. Show that R is a PID.

Solution. Since R is a subring of a field, it is a domain. Let I be an ideal of R. Let $n \geq 0$ be minimial such that $p^n \in R$. Then $(p^n) \subset I$. We claim that we have equality. Indeed, suppose $f/g \in I$ with $f, g \neq 0$. Write $f = p^k f'$ where f' is coprime to p, which is possible since $\mathbf{C}[x,y]$ is a UFD. Then g/f' belongs to R, and so $p^k = (f/g)(g/f')$ belongs to I. By minimality of n, we have $k \geq n$. Thus $f/g = (p^{k-n} f'/g) p^n$ belongs to (p^n), which establishes the claim. Thus every ideal of R is principal. This completes the proof.

Problem 4. Counting up to isomorphism, how many abelian groups G are there such that G is generated by at most three elements and $g^4 = 1$ for all $g \in G$?

Solution. Let G be an abelian group generated by 3 elements such that $g^4 = 1$ for all $g \in G$. First notice that G is finite: indeed, if g_1, g_2, g_3 are generators then every element of G has the form $g_1^{n_1} g_2^{n_2} g_3^{n_3}$ for $n_1, n_2, n_3 \in \{0, \ldots, 3\}$. Also, every element of G has order dividing 4. Applying the structure theorem of finite abelian groups, we have an isomorphism

$$G \cong \mathbf{Z}/2^{e_1}\mathbf{Z} \times \mathbf{Z}/2^{e_2}\mathbf{Z} \times \cdots \times \mathbf{Z}/2^{e_r}\mathbf{Z}$$

for unique positive integers $e_1 \geq \cdots \geq e_r \geq 1$. We must have $e_i \leq 2$ for all i since every element has order ≤ 4. We also must have $r \leq 3$: indeed, $(\mathbf{Z}/2\mathbf{Z})^r$ is a quotient of G, and therefore generated by 3 elements, and thus must have dimension ≤ 3 as a vector space over $\mathbf{Z}/2\mathbf{Z}$ by linear algebra. On the other hand, given any $r \leq 3$ and any sequence of e's, we get a G satisfying the conditions. We can assume $r = 3$ in all cases by allowing some of the e's to be 0. The list of possible e's is then:

$$000, 100, 110, 111, 200, 210, 211, 220, 221, 222$$

We thus see that there are 10 such G's (up to isomorphism).

Problem 5. Let A be a 3×3 integer matrix. Suppose that, considered as a matrix over $\mathbf{C}$, the matrix A has Jordan form

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

Let $\bar{A}$ be the reduction of A modulo p. What are the possible Jordan forms of $\bar{A}$, considered as a matrix over the algebraic closure of $\mathbf{F}_p$?

Solution. Since A is conjugate to a nilpotent matrix, it is nilpotent, and so $\bar{A}$ is nilpotent as well. Thus all of its eigenvalues are 0. There are three possible Jordan forms for a 3×3 nilpotent matrix:

$$\left(\begin{array}{c|c|c} 0 & & \\ \hline & 0 & \\ \hline & & 0 \end{array}\right), \qquad \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Call these I, II, and III.

Case I can occur as the Jordan form of $\bar{A}$: take

$$A = \left(\begin{array}{cc|c} 0 & p & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

Then A has the correct Jordan form over $\mathbf{C}$, but $\bar{A} = 0$. Case II can also obviously occur: simply take

$$A = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

We claim that case III cannot occur. Indeed, let A as in the problem statement be given. Then A has rank 1 over the complex numbers. Thus all 2×2 minors of A vanish. It follows that all 2×2 minors of $\bar{A}$ vanish, and so $\bar{A}$ has rank ≤ 1. But the matrix in case III has rank 2.