May 2017, Qualifying Review Algebra, Morning

Justify all of your answers. We write $\mathbf{C}$, $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ for the complex numbers, the field with p elements, the rational numbers, the real numbers and the integers respectively.

Problem 1. How many isomorphism classes of abelian groups of order 6^4 are there?

Problem 2. Let $\zeta_n = e^{2\pi i/n}$ be a primitive n^{th} root of unity.

(a) For which positive integers n does $\mathbf{Q}(\zeta_n)$ contain $\sqrt{2}$?
(b) For which positive integers n does $\mathbf{Q}(\zeta_n)$ contain $\sqrt[3]{2}$?

Problem 3. Suppose that A and B are complex, invertible $n \times n$ matrices with $AB + BA = 0$. Show that there exists a complex, invertible $n \times n$ matrix C such that $A + CAC = 0$.

Problem 4. Let V be the set of 2×2 real matrices, thought of as a 4-dimensional real vector space. For a real number λ, define a symmetric bilinear form $\langle\, ,\, \rangle$ on V by

$$\langle A, B\rangle = \mathrm{Tr}(AB) + \lambda\, \mathrm{Tr}(AB^t)$$

Here Tr is trace and B^t is the transpose of B. For which λ is this form positive definite?

Problem 5. Let p be a prime number and let n be a positive integer.

(a) Show that there is a positive integer m, depending on p and n, such that if A is an invertible $n \times n$ matrix with entries in $\mathbf{F}_p$ that is diagonalizable over the algebraic closure $\overline{\mathbf{F}}_p$ then $A^m = \mathrm{id}_n$.
(b) Determine the minimal positive m in (a) when $p = 3$ and $n = 4$.

May 2017, Qualifying Review Algebra, Afternoon

Justify all of your answers. We write $\mathbf{C}$, $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ for the complex numbers, the field with p elements, the rational numbers, the real numbers and the integers respectively.

Problem 1. Let G be a finite group and let p be a prime number. Show that the following conditions are equivalent:

(a) The group G acts transitively on a set X such that the cardinality of X is at least 2 and relatively prime to p.

(b) The order of G is not a power of p.

Problem 2. Suppose that R is a commutative ring with 1, and $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of R such that every element of $R \setminus (\mathfrak{p} \cup \mathfrak{q})$ is a unit. Show that at least one of $\mathfrak{p}$ or $\mathfrak{q}$ is maximal.

Problem 3. Suppose that K is a field of characteristic $\neq 2$ and $L = K(\beta)$ is a field extension of K with $\beta^2 + \beta^{-2} \in K$. Show that L/K is a Galois extension.

Problem 4. Suppose that V is a real vector space of dimension n.

(a) Show that there exists a linear map $\varphi \colon \bigwedge^2 V \to \mathrm{Hom}(V^*, V)$ such that

$$\varphi(a \wedge b)(f) = f(a)b - f(b)a$$

for all $a, b \in V$.

(b) Suppose n is odd. Show that no element of the image of φ is invertible.

Problem 5. Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$. A *matching* on V is a set $\{E_1, E_2, E_3, E_4\}$ where each E_i is a two-element subset of V such that $V = E_1 \cup E_2 \cup E_3 \cup E_4$. Let $\mathcal{M}$ be the set of matchings. The group S_8 naturally acts on $\mathcal{M}$, and the action is transitive. Let $G \subset S_8$ be the stabilizer of some matching. How many orbits does G have on $\mathcal{M}$?