QR Exam Algebra
January 3, 2017
Morning

(1) Suppose that R is a commutative ring with 1 with only finitely many ideals. Suppose that $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_d$ are all maximal ideals.
 (a) Show that if $a \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_d$ then a is nilpotent.
 (b) Show that if the number of distinct ideals of R is not a power of 2, then R contains a nonzero nilpotent element.

(2) Suppose that G is group of order $2^4 \cdot 11 \cdot 13 \cdot 17 \cdot 19$ with a normal 2-Sylow subgroup. Show that the center of G contains more than 1 element.

(3) We denote the field with q elements by $\mathbb{F}_q$. Let $\psi : \mathbb{F}_{3^9} \to \mathbb{F}_{3^9}$ be the map defined by $\psi(a) = a^3 - a$. For which positive integers d is the kernel of ψ^d a subfield of $\mathbb{F}_{3^9}$?

(4) Let D_4 be the dihedral group with 8 elements. Construct a Galois extension $K/\mathbb{Q}$ with Galois group D_4. In your example, describe explicitly all intermediate fields L with $\mathbb{Q} \subset L \subset K$ such that $L/\mathbb{Q}$ is an extension of degree 2.

(5) (a) Give an example of a nonzero finitely generated $\mathbb{Z}[X]$-module M which is torsion-free, but not free.
 (b) Give an example of a nonzero finitely generated $\mathbb{Z}[X]$-module M and two irreducible elements $f_1, f_2 \in \mathbb{Z}[X]$ such that $f_1 f_2$ kills M, but M does not decompose as a product $M_1 \times M_2$ such that f_1 kills M_1 and f_2 kills M_2.

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Afternoon

(1) Fix a field k and A be the ring $k[X]/(X^p - 1)$. Classify all simple A-modules in the following two cases:
   (a) $k = \mathbb{Q}$;
   (b) $k = \mathbb{F}_p$, the field with p elements.
   (An A-module M is simple if it has exactly 2 submodules, namely 0 and M itself.)

(2) Let K be a separably closed field, so K does not have any finite seperable field extension other than K itself. Let L/K be a finite nontrivial extension of fields.
   (a) Show that the trace map $\mathrm{Tr} : L \to K$ is the zero map.
   (b) Give an example of such a field extension L/K.

(3) Let V_n be the space of polynomials in x of degree at most n with real coefficients. Define a linear map $\phi : V_n \to V_n$ by $\phi(f) = xf' + f''$. Show that there exists $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and a basis $\{f_0, f_1, \ldots, f_n\}$ of V_n such that $\phi(f_i) = \lambda_i f_i$ for all $i = 0, 1, \ldots, n$.

(4) Suppose that V is a finite dimensional real vector space equipped with a symmetric bilinear form $(\cdot, \cdot)$.
   (a) Show that there exists a bilinear form $(\cdot, \cdot)_\star$ on $\bigwedge^2 V$ with the property

$$(v_1 \wedge v_2, w_1 \wedge w_2)_\star = (v_1, w_1)(v_2, w_2) - (v_1, w_2)(v_2, w_1).$$

   (b) Give the signature of $(\cdot, \cdot)_\star$ in terms of the signature of $(\cdot, \cdot)$.

(5) Show that an abelian group of order 100 cannot act faithfully on a set with 13 elements.