January 2016, Qualifying Review Algebra, Morning

Justify all of your answers. We write $\mathbf{C}$, $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ for the complex numbers, the field with p elements, the rational numbers, the real numbers and the integers respectively.

Problem 1. Let p be a prime number and let $1 \leq n < p^2$ be an integer. Show that every p-Sylow subgroup of S_n is abelian.

Problem 2. Let A be a 5×5 matrix with complex entries. Suppose that the set of all eigenvectors of A, together with the zero vector, forms a two-dimensional subspace of $\mathbf{C}^5$. What are the possible Jordan normal forms of A?

Problem 3. Let p be a prime number and let $d \in \mathbf{F}_p$ be a non-square. Show that the set of matrices of the form

$$\begin{pmatrix} a & b \\ db & a \end{pmatrix}$$

with $a, b \in \mathbf{F}_p$ forms a field (under matrix addition and multiplication).

Problem 4. Let $R = \mathbf{C}[t^2, t^3]$, considered as a subring of $\mathbf{C}[t]$, and let $I \subset R$ be the ideal (t^2, t^3). Compute the dimension of $I \otimes_R R/I$ as a complex vector space.

Problem 5. Suppose that G is a finite group with exactly three conjugacy classes. Show that G is isomorphic to S_3 or $\mathbf{Z}/3\mathbf{Z}$.

January 2016, Qualifying Review Algebra, Afternoon

Justify all of your answers. We write $\mathbf{C}$, $\mathbf{F}_p$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{Z}$ for the complex numbers, the field with p elements, the rational numbers, the real numbers and the integers respectively.

Problem 1. Let A be an $n \times n$ complex matrix. Recall that its characteristic polynomial is defined by $\chi_A(t) = \det(tI - A)$, where I is the identity matrix. Prove the identity

$$\chi_{A^2}(t^2) = \chi_A(t)\chi_{-A}(t).$$

Problem 2. Let G be a finite group of cardinality $2^n m$, with m odd, that contains an element of order 2^n. Show that all order 2 elements of G are conjugate.

Problem 3. Let V be the vector space of 3×3 real matrices. Define a bilinear form $\langle \cdot, \cdot \rangle$ on V by

$$\langle A, B \rangle = \mathrm{trace}(AB - AB^t).$$

Show that $\langle \cdot, \cdot \rangle$ is symmetric and compute its signature.

Problem 4. Let R be a commutative ring with identity such that $IJ = I \cap J$ for all ideals I and J. Show that every prime ideal of R is maximal.

Problem 5. Let $K = \mathbf{Q}(a)$ where a is an algebraic number satisfying $a^2 = 13 + 2\sqrt{13}$. Show that $K/\mathbf{Q}$ is Galois with group $\mathbf{Z}/4\mathbf{Z}$.