AIM Preliminary Exam: Probability and Discrete Mathematics

September 2–4, 2017

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

(a) Let X be a non-negative integer-valued random variable. Show that

$$E[X] = \sum_{n=1}^{\infty} \Pr(X \geq n).$$

(b) Show that $E[X^2] \geq E[X]^2$. Under what conditions do we have equality?

(c) Give examples of discrete random variables X and Y that have the same mean and variance, but different probability mass functions. How far can the mass functions be in ℓ_1 norm, aka total variation norm? (That is, maximize $\sum_a |\Pr(X = a) - \Pr(Y = a)|$.)

<u>**Problem 1**</u>

Problem 1

Problem 2

In an election, suppose Alice gets n votes and Bob gets m votes, where $n > m$. As votes come in, networks report on who is "winning," even though promptness of reporting is highly correlated with outcome in the U.S. For this problem, assume that all orders of votes are equally likely. What is the probability that Alice is always strictly ahead after the first vote is counted?

Hint: Let $P_{n,m}$ be the desired probability. Make a table of $P_{n,m}$ for small values and use induction.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Prove that, of six people, either there is some triple among whom all pairs shake hands or some triple among whom no pair shakes hands.

(b) Prove that the statement is false for five instead of six people.

Problem 3

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Problem 4

In a partially-ordered set with order $\leq$, a **chain** is a set of comparable elements $x_1 \leq x_2 \leq \ldots \leq x_k$ and an **antichain** is a set S of pairwise-incomparable elements: for any $x, y \in S$, we have $x \not\leq y$. Let $(P, \leq)$ be a partially-ordered set one of whose longest chains, C, has length r.

(a) Sketch, as a directed graph, a five-element partial order that is not a total order.

(b) Show that P cannot be partitioned into fewer than r antichains.

(c) Show that P can be partitioned into r antichains.

Problem 4

Problem 5

Suppose you are given a directed graph $G = (V, E)$ with a positive integer capacity c_e on each edge, a designated source $s \in V$, a designated sink $t \in V$, and an integral maximum s-t flow in G given by a flow value f_e on each edge e.

Note: A flow satisfies $0 \leq f_e \leq c_e$ at each edge and, at each vertex v other than s and t, the sum of $\pm f_e$ is zero, where the sum is over edges incident on v and the sign is positive for edges into v and negative for edges out from v.

(a) Now suppose, for one given edge ϵ, we increase c_ϵ to $c'_\epsilon = c_\epsilon + 1$, leaving other $c'_e = c_e$. Show how to find a maximum flow in the c'_e graph in time $|V| + |E|$, assuming an appropriate (common) representation of the graph.

(b) Sketch a graph on 4 vertices in which the new flow is greater than the old and another graph on 4 vertices in which the new flow equals the old.

Problem 5

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