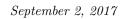
AIM Qualifying Review Exam in Differential Equations & Linear Algebra



There are five (5) problems in this examination.
There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, I sure to mark them clearly and staple them to the booklet.

$\underline{\textbf{Problem 1}}$

(20 points) Consider the matrix
$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 0 & d \\ 1 & 2 \end{pmatrix}$$
 and the vector $\mathbf{b} = \begin{pmatrix} e \\ 0 \\ 5 \end{pmatrix}$. Find values for d and e such that the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has:

- (a) Infinitely many solutions,
- (b) A unique solution,
- (c) No solutions.

Justify your answers.

$\underline{\text{Problem 1}}$

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(20 points) Prove that there exists a real symmetric matrix ${\bf B}$ such that its second power

$$\mathbf{B}^2 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$
 by first showing that the given matrix is positive definite.

How many such **B** are there?

Consider the equation $y'' + y' + \sin y = 0$, describing the motion of a pendulum.

- (a) (8 points) Determine all critical points and their stability.
- (b) (8 points) For each type of critical point, sketch a phase portrait of the solutions in a neighborhood of the critical point in (y, y') space.
- (c) (4 points) How does $y'^2/2 \cos y$ evolve in time? Find a physical interpretation for this behavior, related to the pendulum.

(a) (10 points) Solve the initial value problem

$$y'' - 2y' + y = te^t$$
, $y(0) = 1$, $y'(0) = 1$.

(b) (10 points) Consider the eigenvalue problem $Lu=\lambda u$ with L an operator acting on functions u according to the formula

$$Lu = u'' + u,$$

where u satisfy the boundary conditions

$$u(0) - u'(1) = 0, \ u'(0) - u(1) = 0.$$

Is L a self-adjoint operator on this space of functions?

$\underline{\mathbf{Problem}\ \mathbf{5}}$

(a) (8 points) Find the solution to

$$\begin{split} \partial_{tt} u(x,t) &= \partial_{xx} u(x,t), \ -\infty < x < \infty, \ t > 0 \\ u(x,0) &= x^2 (1-x)^2, \ 0 \le x \le 1, \ u(x,0) = 0 \text{ elsewhere, and } \partial_t u(x,0) \equiv 0. \end{split}$$

Sketch the solution at t = 2.

(b) (12 points) Solve the following equation with given initial and boundary conditions

$$\partial_t w(x,t) - \partial_{xx} w(x,t) = 0, \ 0 \le x \le 1, \ t > 0$$

 $w(x,0) = 0, \ 0 \le x \le 1$
 $w(0,t) = 0, \ w(1,t) = 1, \ t > 0.$

Sketch the solution at positive times $t=t_1\ll 1$ and $t=t_2\gg 1.$

$\underline{ \text{Problem 5}}$

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