AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2022

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Consider a real matrix $\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix}$ where $\mathbf{A}$ is a symmetric square matrix, $\mathbf{B}$ is not necessarily square, and $\mathbf{B}^T$ is the transpose of $\mathbf{B}$.

(a) (8 points) Show that $\mathbf{C}$ is singular if the number of columns of $\mathbf{B}$ is strictly larger than the number of its rows.

(b) (12 points) Show that if $\mathbf{A}$ is strictly positive definite, then $\mathbf{C}$ is nonsingular if and only if the columns of $\mathbf{B}$ are linearly independent.

Solution

(a) Let $\mathbf{B}$ have m rows and n columns, $n > m$. Note that C is $(m+n)$-by-$(m+n)$. Since $n > m$, there is a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ that is not in the span of the rows of $\mathbf{B}$. We can assume $\mathbf{v}$ is orthogonal to each of the rows of $\mathbf{B}$ by subtracting its orthogonal projection on the span of the rows of $\mathbf{B}$. Now form a nonzero vector $\mathbf{w} = [\mathbf{0}; \mathbf{v}] \in \mathbb{R}^{m+n}$ by prepending m zeros to $\mathbf{v}$. $\mathbf{w}$ is a nonzero vector in the null space of $\mathbf{C}$, so $\mathbf{C}$ is singular.

(b) Assume $\mathbf{C}$ is nonsingular. Then the columns of $\mathbf{C}$ are a linearly independent set. Then the columns of $\mathbf{B}$ are a linearly independent set also, which we prove by contradiction. Assume that a nontrivial linear combination of columns of $\mathbf{B}$ equals the zero vector in $\mathbb{R}^m$. It corresponds to a nontrivial linear combination of the last n columns of $\mathbf{C}$, of the form $[\mathbf{B}; \mathbf{0}]$, that equals the zero vector in $\mathbb{R}^{m+n}$, which is a contradiction.

Now assume $\mathbf{C}$ is singular, i.e. $\mathbf{C} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{0}$ with at least one of $\mathbf{x}$ and $\mathbf{y}$ nonzero. If $\mathbf{x} = \mathbf{0}$ we have $\mathbf{B}\mathbf{y} = \mathbf{0}$ with $\mathbf{y}$ nonzero, so the columns of $\mathbf{B}$ are linearly dependent. If $\mathbf{x}$ is nonzero, note that $\mathbf{B}^T\mathbf{x} = \mathbf{0}$ and $\mathbf{A}\mathbf{x} = -\mathbf{B}\mathbf{y}$. Multiplying by $\mathbf{x}^T$ we get $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0$ which contradicts the positive definiteness of $\mathbf{A}$. So $\mathbf{x} = \mathbf{0}$.

Problem 2

Let $\mathbf{A}$ be a 2-by-2 matrix with complex entries that is Hermitian: $\mathbf{A} = \mathbf{A}^*$. Let $\mathbf{a}$ be a column vector, and let $\mathbf{B} = \mathbf{A} + \mathbf{a}\mathbf{a}^*$. Denote the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ by (α_1, α_2) and (β_1, β_2) respectively.

(a) (5 points) Show that $\mathbf{a}\mathbf{a}^*$ is a 2-by-2 matrix that is positive semi-definite. Explain why the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are real.

(b) (10 points) Assume that the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are ordered so that $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Show that $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2$.

(c) (5 points) Find the eigenvalues and eigenvectors of $\mathbf{B}$ if $\alpha_1 = \alpha_2$.

Solution

(a) In order for $\mathbf{A} + \mathbf{a}\mathbf{a}^*$ to make sense, $\mathbf{a}\mathbf{a}^*$ must by 2-by-2. One eigenspace is spanned by $\mathbf{a}$ and has eigenvalue $\mathbf{a}^*\mathbf{a} \geq 0$. The other eigenspace is the one-dimensional subspace $\mathbf{a}^\perp$ of vectors orthogonal to $\mathbf{a}$ and has eigenvalue 0. Since $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, all the eigenvalues are real.

(b) The minimax theorem says α_1 and α_2 are respectively the min and max over $\mathbf{x} \in \mathbb{C}^2$ of $(\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x})$, and likewise for β_1 and β_2 with respect to the same ratio (the Rayleigh quotient) but with $\mathbf{B}$ in place of $\mathbf{A}$. $\beta_1 = \min_{\mathbf{x}} (\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x}) + (\mathbf{a}^*\mathbf{x}, \mathbf{a}^*\mathbf{x})/(\mathbf{x}, \mathbf{x}) \geq \min_{\mathbf{x}} (\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x}) = \alpha_1$. If we choose a nonzero vector $\mathbf{x}_0 \in \mathbf{a}^\perp$, we have $\beta_1 \leq (\mathbf{A}\mathbf{x}_0, \mathbf{x}_0)/(\mathbf{x}_0, \mathbf{x}_0) + (\mathbf{a}^*\mathbf{x_0}, \mathbf{a}^*\mathbf{x_0})/(\mathbf{x_0}, \mathbf{x_0}) = (\mathbf{A}\mathbf{x}_0, \mathbf{x}_0)/(\mathbf{x}_0, \mathbf{x}_0) \leq \max_{\mathbf{x}} (\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x}) = \alpha_2$. Finally, $\beta_2 = \max_{\mathbf{x}} (\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x}) + (\mathbf{a}^*\mathbf{x}, \mathbf{a}^*\mathbf{x})/(\mathbf{x}, \mathbf{x}) \geq \max_{\mathbf{x}} (\mathbf{A}\mathbf{x}, \mathbf{x})/(\mathbf{x}, \mathbf{x}) = \alpha_2$.

(c) If $\alpha_1 = \alpha_2$, $\mathbf{A} = \alpha_1 \mathbf{I}$, a multiple of the identity matrix, so one eigenspace of $\mathbf{B}$ is spanned by $\mathbf{a}$ and has eigenvalue $\alpha_1 + \mathbf{a}^*\mathbf{a}$. The other eigenspace is the one-dimensional subspace $\mathbf{a}^\perp$ and has eigenvalue α_1.

Problem 3

(a) (10 points) Find the first three nonzero terms in each independent series solution about $x = 0$ for the following differential equation

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y = 0.$$

(b) (10 points) Find three independent **real** solutions to the equation

$$x^3\frac{d^3y}{dx^3} + 6y = 0.$$

valid in the domain $x > 0$.

Solution

(a) We plug $y = a_0 + a_1 x + a_2 x^2 + \ldots$ into the equation and satisfy the equation at each power of x. At x^0 we obtain $a_2 = a_0/2$. At x^1 we obtain $a_3 = -a_1/3$. Continuing in this way we find $a_4 = -5a_0/24$ and $a_5 = 2a_1/15$. There are two nontrivial series solutions, one proportional to $1 + x^2/2 - 5x^4/24 + \ldots$ and the other proportional to $x - x^3/3 + 2x^5/15 + \ldots$.

(b) $x = 0$ is a regular singular point, so we try a solution of the form $y = x^r$, and obtain $r^3 - 3r^2 + 2r + 6 = 0$. By trial and error we find a root $r = -1$, and factor the term $r+1$ from the cubic to obtain the quadratic $r^2 - 4r + 6$ with roots $2 \pm \sqrt{2}i$. We have three independent solutions: x^{-1}, $x^{2+\sqrt{2}i}$, $x^{2-\sqrt{2}i}$. The last two need to be manipulated to obtain real solutions. We write $x^{2\pm\sqrt{2}i} = x^2 e^{\pm\sqrt{2}i \log x}$ and Euler's identity gives the two independent real solutions $x^2 \sin(\sqrt{2}\log x)$ and $x^2 \cos(\sqrt{2}\log x)$.

Problem 4

Find the general solution to the system of differential equations

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2+y^2}}(x^2+y^2-4),$$
$$\frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2+y^2}}(x^2+y^2-4).$$

Describe the qualitative behavior of solutions for all initial conditions $(x,y)|_{t=0} = (x_0, y_0) \in \mathbb{R}^2$.

Solution

Using polar coordinates, we have $rdr/dt = xdx/dt + ydy/dt$ so $dr/dt = r^2 - 4$. Thus $r = 2$ is a fixed point of r, r decreases with time for $0 < r < 2$, and r increases with time for $r > 2$. The ODE for r is separable and can be solved using partial fractions. We obtain $r = 2(1+Ke^{4t})/(1-Ke^{4t})$ where $K = (r_0-2)/(r_0+2)$ with $r_0 = r(0)$. The angular coordinate $\theta = \arctan(y/x)$ satisfies $\theta' = (xy' - yx')/r^2 = -1$, so $\theta = \theta_0 - t$. One could now use $x = r\cos\theta$ and $y = r\sin\theta$ to write the solutions in terms of x and y. Hence the circle $r = 2$ is a periodic orbit and trajectories that start on this circle remain on the circle. Initial conditions with $r_0 > 2$ spiral outwards with distance from the origin growing exponentially in time. Initial conditions with $r_0 < 2$ spiral inwards and reach the origin at a time $t = \log(-1/K)/4$. At the origin, the system of ODEs has a singularity, as the limit of $(dx/dt, dy/dt)$ there along a line of constant θ is -2$\hat{\mathbf{e}}_r$, which varies with θ. Therefore, the problem is not well-posed for initial condition at the origin or when the inward spiraling solutions reach the origin.

Problem 5

Solve the PDE

$$\partial_{xx}u + 2\partial_x u + \partial_{yy}u = 0$$

for $u(x,y)$ on the square domain $0 \le x, y \le 1$ with the boundary conditions:

$$u(x,0) = 0 \;;\; u(x,1) = 1 \;;$$
$$u(0,y) = 0 \;;\; u(1,y) = 1.$$

Solution

We decompose the solution as $u = u_1 + u_2$, where u_1 and u_2 satisfy the same PDE but are zero on three sides: $u_1 = 1$ on the $x = 1$ edge and $u_1 = 0$ on the other three sides, and $u_2 = 1$ on the $y = 1$ edge and $u_2 = 0$ on the other three sides. By symmetry, $u_2(x,y) = u_1(y,x)$, so we only need to find u_1 and then we have the solution as $u(x,y) = u_1(x,y) + u_1(y,x)$.

We solve for u_1 using separation of variables: $u_1(x,y) = X(x)Y(y)$. We have

$$\frac{X''}{X} + 2\frac{X'}{X} = -\frac{Y''}{Y} = k^2.$$

for a constant k. Since we have homogeneous boundary conditions at $y = 0$ and 1, we have nontrivial solutions for Y only for $k = n\pi$ with $n = 1, 2, \ldots$; they are $Y(y) = \sin(n\pi y)$. The two corresponding solutions for X are $X(x) = e^{(-1\pm\sqrt{1+n^2\pi^2})x}$. We take the superposition that automatically satisfies $X(0) = 0$: $X(x) = e^{-x}\sinh(\sqrt{1+n^2\pi^2}x)$. So we write $u_1 = \sum_{n=1}^{\infty} a_n e^{-x}\sinh(\sqrt{1+n^2\pi^2}x)\sin(n\pi y)$ and choose the a_n to satisfy $u_1(1,y) = 1$, i.e.

$$\sum a_n e^{-1}\sinh\sqrt{1+n^2\pi^2}\,\sin(n\pi y) = 1.$$

Multiply both sides by $\sin(n\pi y)$ and integrate from 0 to 1 in y to obtain

$$\begin{aligned} a_n &= 4/(n\pi e^{-1} \sinh \sqrt{1+n^2\pi^2}), n \text{ odd}, \\ a_n &= 0, n \text{ even}. \end{aligned}$$

So the final solution is

$$u = \sum_{n=1}^{\infty} a_n \left[e^{-x} \sinh(\sqrt{1+n^2\pi^2}x) \sin(n\pi y) + e^{-y} \sinh(\sqrt{1+n^2\pi^2}y) \sin(n\pi x) \right].$$

with the a_n above.