AIM Qualifying Exam: Advanced Calculus and Complex Variables

August 2022

There are five (5) problems in this test, each worth 20 points.

For the most part, there is sufficient room in this booklet for all your work. However, if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

1. Suppose that $a_0, a_1, a_2, \ldots$ is a sequence of numbers such that

$$\left|\frac{a_{n+1}}{a_n}\right| = 1 - \frac{\alpha}{n+1}$$

for $n = 0, 1, \ldots$ with $\alpha \geq 0$. You may assume the inequality $\log(1-x) \leq -x$ for $-\infty < x < 1$.

a) Begin by unrolling inequalities as in

$$\begin{aligned}\log|a_{n+1}| &\leq \log|a_n| - \frac{\alpha}{n+1} \\ &\leq \log|a_{n-1}| - \frac{\alpha}{n} - \frac{\alpha}{n+1}\end{aligned}$$

and prove that $|a_n| \leq \frac{C}{(n+1)^\alpha}$ for some constant C and $n = 0, 1, 2, \ldots$

b) Write down the nth term of the binomial series

$$(1+z)^{\frac{1}{2}} = 1 + \frac{1}{2}z + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}z^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}z^3 + \cdots$$

and use it prove that the binomial series converges uniformly in the disc $|z| \leq 1$.

2. Suppose $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are sequences of *positive* numbers such that

$$\begin{aligned} a_0 + a_1 + \cdots &= A \\ b_0 + b_1 + \cdots &= B \end{aligned}$$

with $A < \infty$ and $B < \infty$ (both series converge). Define

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &\vdots \end{aligned}$$

and more generally $c_k = \sum_{j=0}^{k} a_j b_{k-j}$. Prove that the series converges to the product AB.

3. Suppose $f(z)$ is analytic for all $z \in \mathbb{C}$ and $|f(z)| = 1$ for $|z| = 1$.

   a) Prove that $|f(z)| \leq 1$ for $|z| \leq 1$.

   b) If $f(z) \neq 0$ for $|z| < 1$, prove that $|f(z)| = 1$ for $|z| \leq 1$.

4. Evaluate the complex integral

$$\int_{-i\infty}^{+i\infty} \frac{e^{z+1}}{z+1}\, dz.$$

The path of integration is iy with y increasing from $-\infty$ to $+\infty$.

5. Let

$$f(z) = \frac{d}{z^2} + \frac{c}{z} + b + z$$

with $d \neq 0$. Suppose $|f(z)| \neq 0$ for $|z| = 1$ and

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = 0,$$

with γ being the circle $|z| = 1$ in the counterclockwise sense. How many roots does the cubic equation $z^3 + bz^2 + cz + d = 0$ have in the region $|z| < 1$? Justify your answer.