AIM Preliminary Exam: Probability and Discrete Mathematics

August 20, 2021

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

For a nonnegative integer-valued random variable N, show that

$$E[N] = \sum_{i=1}^{\infty} P[N \geq i].$$

Problem 1

Problem 1

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Problem 2

Suppose X, Y, Z are i.i.d. real-valued random variables uniformly-distributed on $[0, 1]$.

(a) Sketch the density $f_{X+Y}(x)$ on a Cartesian plane with f_{X+Y} vertical and x horizontal. Label algebraically, where appropriate. Also illustrate (somehow) the joint density for (X, Y) on a Cartesian plane with x horizontal and y vertical. Finally, also illustrate the density of $X + Y$ on this graph with x horizontal and y vertical.

(b) Sketch the density $f_{X+Y+Z}(x)$ on a Cartesian plane with f_{X+Y+Z} vertical and x horizontal. Label algebraically, where appropriate.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Use the binomial theorem to prove that $2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$.

(b) Use combinatorial reasoning to prove, for $n \geq 3$, that

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

Problem 3

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Problem 4

Prove that, if we choose five points in a unit square, some pair of the points are at distance at most $1/\sqrt{2}$.

Problem 4

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Problem 5

Let G be an undirected graph with distinct edge weights. Let S be a subset of the vertex set V that is neither empty nor V. Let $e = (v, w)$ be the minimum-cost edge with $v \in S$ and $w \in V \setminus S$.

Prove that every minimum spanning tree contains e.

Problem 5

Problem 5

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