AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2021

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $\mathbf{A} = \begin{bmatrix} 1 & c \\ d & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consider the problem of minimizing $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$ over $\mathbf{x} \in \mathbb{R}^2$.

(a) For which values of c and d is there a unique solution for $\mathbf{x}$?

(b) When there is not a unique solution, express the set of $\mathbf{x}$ that achieve the minimum possible value of $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$ in terms of only one of c or d, but not both.

Problem 1

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Problem 2

(a) Let V and W be two linear subspaces of $\mathbb{R}^n$. Prove or disprove: Both the union and intersection of V and W are linear subspaces of $\mathbb{R}^n$.

(b) What is the dimension of the set of vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ that are simultaneously solutions of

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ?$$

Problem 2

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Problem 2

Problem 3

Find a constant C such that the differential equation

$$\frac{d^2y}{dx^2} + y = C \cos 2x + x^3 + \sin x$$

has a solution that obeys

$$y(0) = y(\pi) = 0.$$

Problem 3

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Problem 3

Problem 4

(a) Find the general solution to the linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}.$$

(b) Identify the terms in your solution that have the largest magnitudes in the limits $t \to +\infty$ and $t \to -\infty$.

Problem 4

Problem 4

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Problem 5

Solve the PDE

$$\partial_{tt}u - \partial_{xx}u = 0$$

for $u(x,t)$ on the interval $0 \leq x \leq \pi$ for $t > 0$ with the initial and boundary conditions:

$$\begin{aligned} u(x,0) = 1\ ,\ \partial_t u(x,0) = 0\ ,\ 0 \leq x \leq \pi \\ u(0,t) = u(1,t) = 0\ ,\ t > 0. \end{aligned}$$

Problem 5

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