AIM Preliminary Exam: Probability & Discrete Mathematics

September 4, 2011

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Suppose X_1 and X_2 are independent normal random variables with mean zero and variance 1. Define

$$Y := \frac{X_1}{1 + |X_2|}.$$

(a) What is $\mathbb{E}[Y]$? Justify your answer.

(b) Express the distribution function of Y in terms of multiple integrals.

Problem 1

Problem 1

Problem 1

Problem 2

A subset of nodes $S \subset V$ is an *independent set* of a graph $G = (V, E)$ if there are no edges between the nodes. In Figure 1, the nodes $\{1, 5\}$ form an independent set, but nodes $\{1, 4, 5\}$ do not, because of the edge between 4 and 5. One of several largest independent sets is $\{2, 3, 6\}$.

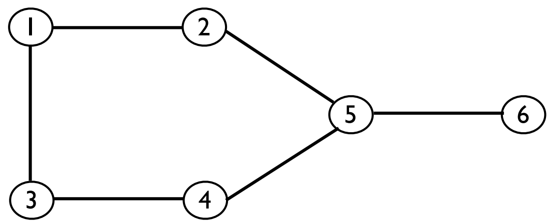

Figure 1: Graph $G = (V, E)$ with numbered nodes.

(a) Devise a deterministic, polynomial time algorithm that, given a graph $G = (V, E)$ which is a *tree* with root r, computes the size of the largest independent set in G. Prove that your algorithm computes the desired quantity. Hint: use dynamic programming.

(b) What is the running time, in terms of $|V|$ and $|E|$ of your algorithm?

Problem 2

Problem 2

Problem 2

Problem 3

Let $G = (V, E)$ be a graph on n vertices and m edges such that $m \geq n - 1$. The following randomized algorithm produces an independent set with at least $n^2/(4m)$ vertices. See Problem 3 for the definition of an independent set.
Let $d = 2m/n$ be the average degree of the vertices in G.

1. Delete each vertex of G (together with its incident edges) independently with probability $1 - 1/d$.
2. For each remaining edge, remove it and one of its adjacent vertices.

The algorithm then terminates, returning the set of remaining vertices.

(a) Why do the remaining vertices form an independent set?

(b) Let X be the number of vertices that survive the first step of the algorithm. Compute $\mathbb{E}[X]$ and express it in terms of n, m, and/or d.

(c) Let Y be the number of edges that survive the first step of the algorithm. Compute $\mathbb{E}[Y]$ and express it in terms of n, m, and/or d.

(d) Let Z be the size of the independent set. What is Z in terms of X and Y? Compute $\mathbb{E}[Z]$.

Problem 3

Problem 3

Problem 3

Problem 4

Let $X_1, \ldots, X_n$ be independent uniform random variables over $[0, 1]$. Let $Y_1, \ldots, Y_n$ be the same values as $X_1, \ldots, X_n$ sorted in increasing order.

(a) Show that $\mathbb{E}[Y_1] = 1/(n + 1)$; *i.e.*, show that the expected value of the *smallest* of $X_1, \ldots, X_n$ is $1/(n + 1)$.

(b) Extend your argument from part (a) to show that $\mathbb{E}[Y_k] = k/(n + 1)$.

Problem 4

Problem 4

Problem 4

Problem 5

A collection of discrete random variables $X_1, \dots, X_n$ is said to be *pairwise independent* if for any pair j and k ($j \neq k$) and any values a and b

$$\mathbb{P}[(X_j = a) \cap (X_k = b)] = \mathbb{P}[X_j = a]\mathbb{P}[X_k = b].$$

Let p be a prime number and choose two independent, uniformly distributed values X_1 and X_2 over $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$. Set

$$Y_j = X_1 + jX_2 \mod p \quad \text{for } j = 0, \dots, p-1.$$

(a) Show that the random variables $Y_0, \dots, Y_{p-1}$ are uniformly distributed random variables over $\mathbb{Z}_p$. Hint: use conditional probability.

(b) Show that the random variables $Y_0, \dots, Y_{p-1}$ are pairwise independent random variables over $\mathbb{Z}_p$. Hint: Consider the event $E_{j,k}(a, b)$ that

$$Y_j = a \quad \text{and} \quad Y_k = b$$

for $j \neq k$. Set up and then solve a system of equations implied by the event $E_{j,k}(a, b)$ for the random variables X_1 and X_2.

(c) Let $Y = \sum_{j=0}^{p-1} Y_j$. What is $\mathrm{Var}[Y]$?

(d) Derive an upper bound for

$$\mathbb{P}[|Y - \mathbb{E}[Y]| \geq a]$$

using your expression for $\mathrm{Var}[Y]$ from part (c).

Problem 5

Problem 5

Problem 5