AIM Preliminary Exam: Differential Equations & Linear Algebra

January 7, 2013

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Consider the system of two *second-order* ordinary differential equations for functions $x(t), y(t)$,

$$x'' + x = ay, \quad y'' + y = bx,$$

where a, b are real constants. The system is *stable* if all solutions are either bounded or tend to zero as $t \to \infty$; otherwise the system is *unstable*. Determine the regions in the ab-plane in which the system is stable and unstable. Sketch the regions and indicate their stability.

Problem 1

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Problem 2

Consider the heat equation with a source term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u,$$

on the domain $\{(x,t) : 0 < x < \pi, t > 0\}$, subject to boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(\pi,t) = 0,$$

and initial condition

$$u(x,0) = 2\cos^2\left(\frac{x}{4}\right) - 2\cos^2\left(\frac{3x}{4}\right).$$

(a) Find $u(x,t)$.

(b) Sketch the graph of $u(x,t)$ on the interval $0 \leq x \leq \pi$ for large t.

Problem 2

Problem 2

Problem 2

Problem 3

Let $u(x, y)$ be the solution of the Laplace equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

on the rectangle $\{(x, y) : 0 \le x \le 2a, 0 \le y \le 2b\}$, with $a > 0$ and $b > 0$, and Dirichlet boundary conditions,

$$u(x, 0) = u(x, 2b) = u(0, y) = 0, \quad u(2a, y) = 1.$$

Show that the value of the solution at the center of the rectangle satisfies

$$\left| u(a, b) - \frac{2}{\pi} \operatorname{sech}\left(\frac{\pi a}{2b}\right) \right| < \frac{2}{3\pi} \operatorname{sech}\left(\frac{3\pi a}{2b}\right).$$

Problem 3

Problem 3

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Problem 4

Determine whether each of the following statements is true or false. Justify your answer in each case.

(a) If a linear system $Ax = b$ has two distinct solutions, then it has infinitely many distinct solutions.

(b) Similar matrices have the same eigenvalues and eigenvectors.

(c) If $A^3 = 0$, then $I + A$ is invertible.

(d) If A is a real symmetric matrix, then there exists another real symmetric matrix B such that $A = B^3$.

(e) The vectors $q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $q_2 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$, $q_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $q_4 = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$ are linearly independent over the complex numbers.

Problem 4

Problem 4

Problem 4

Problem 5

Given a three-component real column vector v, define a sequence of real scalars $\{\alpha_k\}_{k=1}^{\infty}$ by the relation

$$\alpha_k := \frac{v^T A^k v}{v^T A^{k-1} v}, \quad \text{where} \quad A := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and where as usual v^T denotes the row vector transpose of v. Find $\lim_{k \to \infty} \alpha_k$ in each case below.

(a) $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Problem 5

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