AIM Preliminary Exam: Differential Equations & Linear Algebra

January 7, 2012

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let $\mathbf{A}$ be a real, skew-symmetric, $n \times n$ matrix. Show that

$$\det(\mathbf{A} - \mathbb{I}) > 0 \text{ for } n \text{ even,}$$

and

$$\det(\mathbf{A} - \mathbb{I}) < 0 \text{ for } n \text{ odd.}$$

Problem 1

Problem 1

Problem 1

Problem 2

Let $\mathbf{A}$ be an $n \times n$, real, upper triangular matrix with positive diagonal elements. Prove that there is an $n \times n$, real, upper triangular matrix $\mathbf{B}$ with positive diagonal elements such that

$$\mathbf{B}^2 = \mathbf{A}.$$

Hint: Induction on n.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Consider the system of ordinary differential equations

$$\frac{dx}{dt} = x\left(\frac{3}{2} - \frac{1}{2}y\right)$$
$$\frac{dy}{dt} = y\left(-\frac{1}{2} + x\right)$$

For (x, y) in the closed first quadrant: find and classify all critical points, accurately sketch the phase portrait, and describe the behavior of all solutions.

(b) Consider the following differential equation for $y(x)$ for $x > 0$:

$$xy'' - y' + 4x^3 y = 0$$

It has $y_1(x) = \sin(x^2)$ as a solution. Find another linearly independent solution for $x > 0$.

Problem 3

Problem 3

Problem 3

Problem 4

Consider the system of ordinary differential equations

$$\begin{aligned} x'(t) &= xy - x^2y + y^3 \\ y'(t) &= y^2 + x^3 - xy^2 \end{aligned}$$

Find all critical points and determine their stability, sketch the phase portrait in a neighborhood of each critical point, and describe the asymptotic behavior of all solutions.

Hint: Polar coordinates.

Problem 4

Problem 4

Problem 4

Problem 5

Solve the following boundary value problem in the infinite strip $D = \{x > 0\} \times \{0 < y < 1\}$:

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in D,$$

subject to the boundary conditions:

$$u(0, y) = 10y, \quad 0 < y < 1,$$

$$u(x, 0) = 20, \quad x > 0,$$
$$u(x, 1) = 50, \quad x > 0,$$

so that $u(x, y)$ is bounded on D.

<u>Problem 5</u>

Problem 5

Problem 5