AIM Preliminary Exam: Advanced Calculus & Complex Variables

January 7, 2013

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Use contour integrals of the integrand $z^{-2}\cot(z)$ along appropriate square paths centered at the origin to find the value of the infinite series
$$S := \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Justify any limit you need to take.

<u>**Problem 1**</u>

Problem 1

<u>**Problem 1**</u>

Problem 2

Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of continuously differentiable functions $f_n : [0,1] \to \mathbb{R}$ converging pointwise on $[0,1]$ to f, and that there is a constant $M > 0$ such that $|f_n'(x)| \le M$ holds for all $x \in [0,1]$ and all $n \ge 1$.

(a) Prove that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the sense of *uniform* convergence on $[0,1]$. Hint: consider a suitable discrete grid of points in $[0,1]$.

(b) Since each f_n is in particular continuous, part (a) implies that the limit function f has to be continuous on $[0,1]$ (do not prove this). Give an example of a sequence $\{f_n\}_{n=1}^\infty$ satisfying all of the hypotheses for which the limit function is, however, not differentiable. Sketch graphs of a few of the f_n's and the limit function f.

Problem 2

Problem 2

Problem 2

Problem 3

In a certain global climate model, the atmosphere is divided up into cubic cells of side-length L km that are aligned with the (x, y, z)-coordinate axes. In the neighborhood of one of these cells the flow rate of CO_2 is observed at some moment of time, and at position (x, y, z) measured in km with origin at the center of the cell, the rate is $\phi_0 \Phi(x, y, z)$, where

$$\Phi(x, y, z) := \sqrt{(x/L)^2 e^{2y/L} \sin^4(y/L) + (y/L - e^{y/L})^2 + (z/L)^2 e^{2y/L} \cos^4(y/L)}.$$

Here, ϕ_0 is a constant with units of kg per square km of cross-sectional area per hour, and the flow is in the direction

$$\vec{v}(x, y, z) := \frac{1}{\Phi(x, y, z)} \begin{pmatrix} (x/L)e^{y/L} \sin^2(y/L) \\ y/L - e^{y/L} \\ (z/L)e^{y/L} \cos^2(y/L) \end{pmatrix}.$$

(a) Assuming no sources or sinks of CO_2, find the rate of change of the total mass of CO_2 contained in the cell.

(b) Rederive your answer from part (a) by a different method.

Problem 3

Problem 3

Problem 3

Problem 4

(a) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two complex sequences. Evaluate in closed form:

$$\sum_{n=1}^{N} a_n(b_n - b_{n-1}) + \sum_{n=1}^{N} (a_n - a_{n-1})b_{n-1}$$

for arbitrary N.

(b) Prove the following generalization of the alternating series convergence test: let $\{d_n\}_{n=0}^{\infty}$ be a complex sequence for which the partial sums

$$s_N := \sum_{n=0}^{N} d_n$$

satisfy $|s_N| \leq M$ for some constant $M > 0$ and for all $N \geq 0$, and let $\{c_n\}_{n=0}^{\infty}$ be a positive non-increasing sequence for which $\lim_{n\to\infty} c_n = 0$. Then the infinite series

$$\sum_{n=0}^{\infty} c_n d_n$$

converges. (The usual alternating series test is the special case of $d_n = (-1)^n$.) Hint: express d_n in terms of the partial sums.

Problem 4

Problem 4

Problem 4

Problem 5

Find the maximum and minimum values of the function $f(x, y, z) = x + y + z$ taken over the part of the ellipsoidal surface

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{6} = 12$$

with $x \geq 0$.

Problem 5

Problem 5

Problem 5