AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

January, 2017

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1 Let $a \in \mathbb{R}$, and consider the series

$$\sum_{n=0}^{\infty} \sin\left(\pi(a^2 + n^2)^{1/2}\right).$$

Find the values of a for which the series converges, and the values for which it diverges. Justify your answer.

Problem 1

Problem 1

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Problem 2

Let z_1, z_2, z_3, z_4 be the four vertices of a square in $\mathbb{C}$. The sides of the square are *not* necessarily parallel to the axes. Let $p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$. Prove that all three roots of $p'(z) = 0$ are inside the square. *Hint:* Start by working out the problem for a particularly simple square.

Problem 2

Problem 2

Problem 2

Problem 3

Evaluate the integral

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx.$$

Problem 3

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Problem 4 Evaluate the double integral

$$\int_0^1 \int_0^1 \frac{1}{(1+x^2+y^2)^{3/2}} \, dx \, dy.$$

Problem 4

Problem 4

Problem 5 Consider the line integral

$$\int_C y^3 \, dx + (3x - x^3) \, dy,$$

where C is assumed to be a simple closed curve with positive (or counterclockwise) orientation. Find C that maximizes the line integral as well as the maximum value of the integral.

Problem 5

Problem 5

Problem 5