AIM Qualifying Review Exam: Advanced Calculus & Complex Variables

August 30, 2014

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Let R be the region in the plane bounded by the lines $x+y=1$, $x+y=4$, and the coordinate axes. Evaluate the integral

$$\iint_R \frac{y}{(x+y)^2}\,dx\,dy$$

by using the change of variables

$$u = x + y \text{ and } v = \frac{y}{x+y}.$$

Sketch the domain of integration before and after the coordinate change.

Problem 1

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Problem 2

Show that if the sequence $\{x_n\}_{n=0}^{\infty} \subset \mathbb{R}$ satisfies

$$\lim_{n\to\infty} x_n = x$$

then

$$\lim_{p\to 1^-} (1-p)\sum_{n=0}^{\infty} x_n p^n = x$$

where $p \to 1^-$ means p approaches 1 from the left. Next, show that the converse is in general false.

Problem 2

Problem 2

Problem 2

Problem 3

(a) Let $f(x)$ be a twice differentiable function on $\mathbb{R}$. Show that for any $h > 0$ and any $x \in \mathbb{R}$ there exists a ξ between $x - h$ and $x + h$ such that

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(\xi).$$

(b) Let f be a differentiable function on $\mathbb{R}$. Show that for any $k \in \mathbb{R}$, the function

$$f' + kf$$

has a root between any two distinct roots of f.

Problem 3

Problem 3

Problem 3

Problem 4

Evaluate the following integral:

$$\int_0^\infty \frac{1}{1+x^n}\,dx$$

for all positive integers $n \geq 2$, using complex variable methods. Sketch any contours you may use, and indicate where the singularities of your integrand are.

Problem 4

Problem 4

Problem 4

Problem 5

How many roots, counting multiplicity, does the polynomial

$$p(z) = z^5 + 3z + 1$$

have in the first quadrant $\{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0\}$? Give a full explanation for your answer.

Problem 5

Problem 5

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