AIM Qualifying Review Exam in Advanced Calculus & Complex Analysis

August 31, 2019

There are five (5) problems in this examination, each worth 20 points.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Part (b) is a discrete version of part (a).

1. (10 points) Suppose $f(x) \geq 0$ for $x \geq 0$, with $\int_0^\infty f(x)\,dx < \infty$, $\int_0^\infty xf(x)\,dx < \infty$, and f a continuous function. Define $F(x) = \int_x^\infty f(u)\,du$. Prove that

$$\int_0^\infty xf(x)\,dx = \int_0^\infty F(x)\,dx.$$

2. (10 points) Suppose $a_i \geq 0$ for $i = 1, 2, 3, \ldots$ and $\sum_{i=1}^\infty ia_i < \infty$. Define $A_i = a_i + a_{i+1} + a_{i+2} + \cdots$. Prove that

$$\sum_{i=1}^\infty ia_i = \sum_{i=1}^\infty A_i.$$

Problem 1

Problem 1

Problem 1

Problem 2

Evaluate

$$\int_0^\infty \frac{\cos x - e^{-x}}{x}\, dx.$$

Problem 2

Problem 2

Problem 2

Problem 3

Assume a positive and $a > 1$. Evaluate the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - a| \, d\theta.$$

Problem 3

Problem 3

Problem 3

Problem 4

Find an analytic function $f(z)$ which maps the unit disk $\{z \,|\, |z| < 1\}$ to the upper half of the complex plan $\{w \,|\, \mathrm{Im}(w) > 0\}$ bijectively.

Problem 4

Problem 4

Problem 4

Problem 5 The surface of the unit sphere S_2 given by $x^2 + y^2 + z^2 = 1$ is parametrized using spherical coordinates as

$$\mathbf{r} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}.$$

1. (5 points) Find the scalar surface area element dS in spherical coordinates and use it to evaluate

$$\int_{S_2} z^2\,dS.$$

2. (15 points) Evaluate

$$\int_{S_2} \frac{dS}{((x-a)^2 + y^2 + z^2)^{1/2}}$$

for $a > 1$.

Problem 5

Problem 5

Problem 5