42ND UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION

1pm-4pm, April 5, 2025, final

Problem 1. Determine the value of the following limit or prove that it does not exist:

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt[4]{2} + \sqrt[6]{6} + \sqrt[8]{24} + \cdots + \sqrt[2n]{n!}}{n^{3/2}}.$$

Problem 2. Determine the maximum number of pairwise incongruent right triangles having integer edges (a, b, c) with $1 \le a \le b \le c$ with at least one edge having squared edge-length 2025.

Problem 3. Show that

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{k^3}{k!}}{\displaystyle\sum_{k=0}^{\infty} \frac{k^2}{k!}}$$

is a rational number.

Problem 4. Let $*$ be a binary operation on a set A.

(a) Suppose that $(a * b) * c = (a * c) * b$ for all a, b, $c \in A$ and that $*$ has a left identity element e, i.e. an element e such that $e * a = a$ for all a in A. Prove that $*$ is commutative and associative.

(b) Suppose instead that $(a * b) * c = (a * c) * b$ and that $a * (b * c) = c * (b * a)$ for all $a, b, c \in A$. Also assume that $*$ has a right identity element f such that $a * f = a$ for all $a \in A$. Must $*$ be commutative? Must $*$ be associative? Prove your answer.

Problem 5. Let $b > 1$ be a real number. A point $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is chosen at random in the n-dimensional cube C_n defined by the conditions $r_i \in [1, b]$, $1 \le i \le n$. That is, $C_n = [1, b] \times \cdots \times [1, b] = [1, b]^n$. Assume that if two regions within C_n have the same n-volume, then the point is as likely to be in one as in the other. Let E_n be the expected value of the geometric mean $(\prod_{i=1}^{n} r_i)^{1/n}$ of the n coordinates of $\mathbf{r}$. Find $\lim_{n\to\infty} E_n$ as a function $f(b)$ of b, for $b > 1$.

Problem 6. Define a *special sequence* of vectors $\mathbf{v}$ in $V = \mathbb{C}^4$ (viewed as complex-valued 4×1 column vectors) to be a sequence $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ with the two properties:

(a) $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ span V.

(b) There exists a linear transformation T on V such that $T\mathbf{v_i} = \mathbf{v_{i+1}}$ for $1 \le i \le 4$ and $T\mathbf{v_5} = \mathbf{v_1}$.

Show that if $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ are the first four terms of some special sequence, then there are exactly five distinct choices of a vector $\mathbf{v_5}$ in V so that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ is a special sequence.

Problem 7. Let S be the smallest set of positive integers that satisfies the following three conditions:

(1) $4 \in S$.

(2) If $a \in S$, then $a^2 \in S$.

(3) If $a, b \in S$ and $a + b$ is even, then $\frac{a+b}{2}$ is in S.

Determine, with proof, the least integer $n \ge 2025$ that is a member of S.

Problem 8. Let $z_1, z_2, \ldots, z_n$ be arbitrary complex numbers. Show that there is a subset $I \subseteq \{1, 2, \ldots, n\}$ such that

$$\left| \sum_{j \in I} z_j \right| \ge \frac{1}{\pi} \sum_{j=1}^{n} |z_j|.$$

Problem 9. Let $P(z)$ be a polynomial with real coefficients, and set

$$Q(z) = \frac{P(z+i) - P(z-i)}{2i}.$$

Show that if all the zeros of P are real, then all the zeros of Q are real.

Problem 10. Find all functions $f : \{0, 1, ..., 100\} \mapsto \{0, 1, ..., 100\}$ satisfying:

$$f(f(n)) = n - 1 \quad \text{for} \quad 1 \le n \le 99,$$

and

$$f(f(100)) = 0.$$

(*Note.* The value $f(f(0))$ is not specified.)