Subgroups of discrete reflection groups

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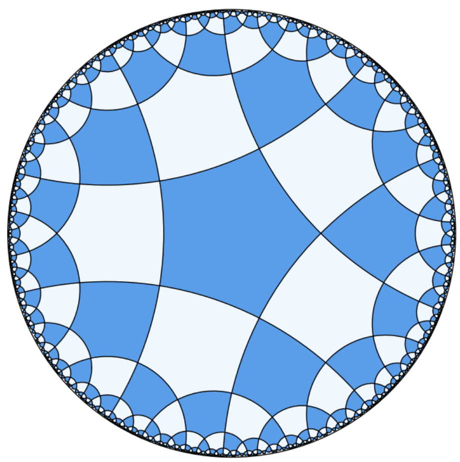
A *reflection group* is a discrete group of symmetries of some space X generated by *reflections*: involutions of X which fix a unique hyperplane $W \subset X$ (a "wall" or "mirror"), and exchange the two half-spaces on either side of W. Reflection groups abound throughout geometry, and include basic examples like dihedral groups and finite symmetric groups, as well as more complicated groups that generate beautiful tilings of non-Euclidean spaces (for instance hyperbolic space).



Reflection groups are very well-studied partly because of their ubiquity, but also because it's possible to get a very concrete understanding of their structure through the abstract theory of *Coxeter groups*. Possibly the easiest Coxeter groups to understand are the *right-angled Coxeter groups*, which correspond to reflection groups where the "walls" of the reflections meet at right-angles. Despite their relatively simple definition, right-angled Coxeter groups (and their cousins, the right-angled Artin groups) have a rich theory—for instance, they played a key role in the celebrated proof of the virtual Haken and virtual fibering conjectures, deep structural results about the topology of 3-dimensional manifolds.

We will spend a lot of this project learning about the theory of right-angled Coxeter groups, and the way they can be realized as groups of matrices in $\mathrm{GL}(n, \mathbb{Z})$. Depending on time and interest, we may also explore the connection to cube complexes and right-angled Artin groups. Ultimately, the goal would be to try and find interesting *subgroups* of certain right-angled Coxeter groups. Specifically, we will be looking for matrix subgroups which have the *Anosov property*, meaning that the singular value decompositions of elements of the subgroup satisfy a particular exponential growth condition. Anosov subgroups are interesting to researchers in a number of different areas, but actual constructions can be elusive—so the long-term aim of this project would be to provide evidence that right-angled Coxeter groups give a good way to find a wide variety of them.

Prerequisites

- Should be very comfortable with linear algebra (we may have to spend some time learning linear algebra topics beyond what is covered in Math 217)
- Group theory (Math 412 or 493 or equivalent)
- Some programming experience (Python would be ideal, but C/C++, Matlab, Mathematica, etc. all fine)
- Useful, but not required: knowledge of real projective space and basic algebraic topology (fundamental groups, covering spaces, cell complexes).