# Lengths of Quotient Rings

Jenny Kenkel

What is the vector space dimension of the set of all polynomials with variable $x$ of degree $d$ or less? What about the vector space dimension of the set of all polynomials with variables $x_1$ and $x_2$, such that $x_1$ appears with exponent no more than $d_1$, and $x_2$ appears with exponent no more than $d_2$? What if we have $n$ variables, $x_1$ through $x_n$, such that $x_i$ appears with exponent no more than $d_i$ for $i$ from 1 to $n$? We can investigate this question by studying the number of **monomials** (that is, elements of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$) in the quotient ring

$$k[x_1, \dots, x_n]/(x_1^{n_1}, \dots, x_n^{d_n})$$

where $k$ is some field. Does your answer change if $k$ is a field of positive characteristic, that is, something like $\mathbb{Z}_p$ for some prime number $p$?

This question has an answer that is relatively straightforward to write down, but adding in just one relation, and instead studying the quotient ring

$$k[x_1, x_2, x_3]/(x_1^{n_1}, x_2^{d_2}, x_3^{d_3}, x_1 + x_2 + x_3)$$

where $k$ is a field of positive characteristic is a much harder questions! This question has been studied in various works: [RRR91, Han92, HM93, Vra15]. While it is a large problem, to find a closed form equation, it's straightforward to calculate examples via algebra software Macaulay2. We will begin by restricting our attention to a field of characteristic 2, generating many examples using algebra software Macaulay2 and case-by-case investigations. We'll compare these lengths to well-known sequences using the Online Encyclopedia of Integer Sequences [Inc22], and then form conjectures for lengths or bounds that we will then prove. Depending on our success, we may generalize to more variables or other characteristics.
**Prerequisites:** Math 412 or 493 (or equivalent).

## References


[Han92] C. Han. The Hilbert-Kunz function of a diagonal hypersurface. ProQuest LLC, Ann Arbor, MI, 1992. Thesis (Ph.D.)–Brandeis University.

[HM93] C. Han and P. Monsky. Some surprising Hilbert-Kunz functions. Math. Z., 214(1):119–135, 1993.

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[RRR91] L. Reid, L. G. Roberts, and M. Roitman. On complete intersections and their Hilbert functions. Canad. Math. Bull., 34(4):525–535, 1991.

[Vra15] A. Vraciu. On the degrees of relations on $x_1^{d_1}, \dots, x_n^{d_n}, (x_1 + \cdots + x_n)^{d_{n+1}}$ in positive characteristic. J. Algebra, 423:916–949, 2015.