Symplectic Linear Algebra of Semi–Clifford Quantum Gates

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Overview

Quantum computing uses quantum-mechanical phenomena to perform computational tasks that may be intractable for classical computers. Mathematically, a quantum computation is modeled as a sequence of quantum gates. A fundamental challenge, however, is that quantum information is extremely sensitive to noise and errors arising in the physical implementation of a quantum computer. Quantum error correction addresses this problem by encoding quantum information in a way that allows such errors to be detected and corrected [1, 2]. An important aspect of quantum error correction is understanding how noise affects a computation, which requires determining how errors propagate through quantum gates. Basic quantum errors are represented by Pauli operators, which together form the Pauli group. If a Pauli error P occurs before a quantum gate U, then the corresponding error after the gate is $UPU^\dagger$. Thus, conjugation describes how a quantum gate transforms an error as it passes through a quantum circuit.

Clifford Hierarchy

The Clifford hierarchy [3] formalizes this conjugation-based description of errors by organizing quantum gates according to the complexity of the quantum gates obtained under conjugation. The first level is the Pauli group, while the second level consists of Clifford gates, which map Pauli operators to Pauli operators under conjugation. It is known that quantum circuits composed of Clifford gates can be simulated efficiently on a classical computer [4]. The third level contains more general non-Clifford gates that, together with Clifford gates, enable universal quantum computation. In particular, a gate U in the third level is called *semi-Clifford* if $U = C_1 D C_2$, where $C_1, C_2 \in \mathcal{C}_2$ are Clifford gates and D is a diagonal gate in the third level. Semi-Clifford gates can admit more efficient implementations in quantum circuits.

Project Description

The purpose of this project is to study semi-Clifford gates in the third level of the Clifford hierarchy using Python and SageMath. This computational problem is motivated by ongoing work [5] that associates each third-level gate U with a subspace K_U of a symplectic vector space. The gate U is semi-Clifford precisely when K_U contains a special type of subspace called a Lagrangian subspace. The existence of such a subspace therefore provides a concrete certificate that U is semi-Clifford. The project will investigate these certificates through the following objectives:

1. determine whether K_U contains a Lagrangian subspace and construct one explicitly when it exists;

2. compute the relevant subspaces exhaustively for small quantum systems;

3. develop algorithms for larger systems and test them on structured families of third-level gates;

4. determine whether different certificates yield inequivalent Clifford–diagonal–Clifford decompositions;
5. investigate whether an appropriate choice of certificate can reduce the complexity of describing or implementing a semi-Clifford gate.

Disclaimer

Terms such as *group*, *conjugation*, *adjoint*, the notation $U^\dagger$, *Clifford hierarchy*, *symplectic vector space*, and *Lagrangian subspace* may be unfamiliar at first. This should not be a concern, as no prior familiarity with this terminology is required. In fact, the relevant definitions, examples, and methods will be introduced as they are needed, and students will become comfortable working with these concepts over the course of the project.

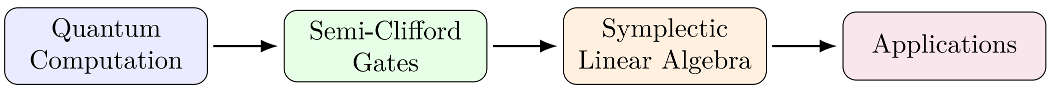

Figure 1: Schematic description of the project

Relation to Existing Work

Semi-Clifford gates were introduced by Zeng et al. [6]. De Silva [7] analyzed semi-Clifford gates using symplectic linear algebra methods and developed an algorithm for identifying the relevant subspaces. More recent work has studied these gates from the perspective of algebraic geometry [8].

Prerequisites

There are no formal prerequisites. Prior coursework in linear algebra and abstract algebra, together with some experience with Python or SageMath, would be helpful but is not required. The necessary background in quantum computing and symplectic linear algebra can be learned as needed during the project.

References

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[4] Daniel Gottesman. *The Heisenberg Representation of Quantum Computers*. 1998. arXiv: `quant-ph/9807006 [quant-ph]`.

[5] Junaid Aftab. “The Clifford Hierarchy via Quantum Derivatives”. In preparation. 2026.

[6] Bei Zeng, Xie Chen, and Isaac L. Chuang. “Semi-Clifford operations, structure of $\mathcal{C}^k$ hierarchy, and gate complexity for fault-tolerant quantum computation”. In: *Physical Review A* 77.4 (2008), p. 042313.

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[8] Imin Chen and Nadish de Silva. “Characterising semi-Clifford gates using algebraic sets”. In: *Communications in Mathematical Physics* 405.9 (2024), p. 201.