Isospectral lattices

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A *lattice* is a grid of regularly repeating points: more precisely, a lattice can be constructed by taking a basis for $\mathbb{R}^n$ and considering the set of all linear combinations of these basis vectors that have *integer* coefficients. Lattices can come in many different shapes and sizes, and as the dimension grows, they can become incredibly difficult to study: so hard, in fact, that the security of many modern cryptographic protocols depend on the infeasibility of finding short vectors in high-dimensional lattices.

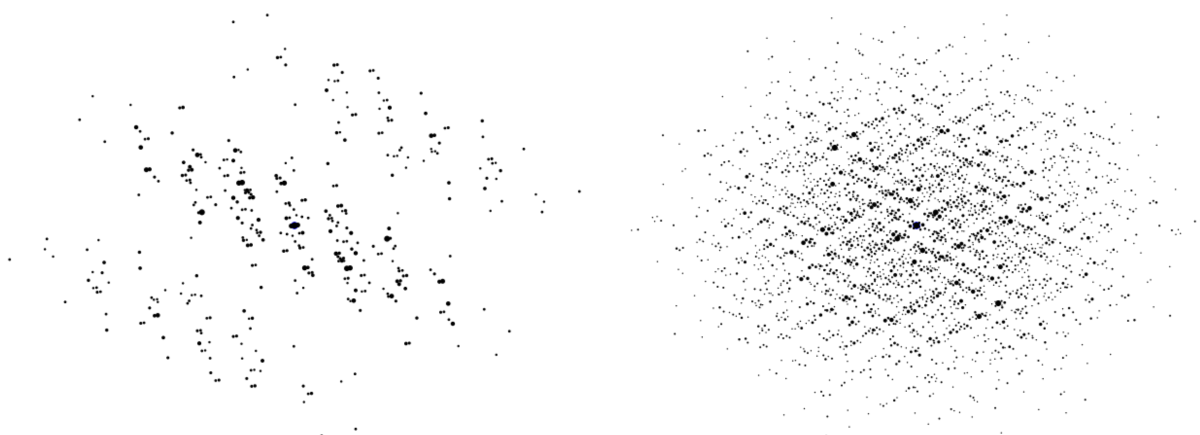

Figure 1: Examples of 10-dimensional lattices projected into two dimensions

One way to identify a lattice is to record the number of vectors of each length. For example, the cubic lattice $\mathbb{Z}^3$ has one vector of length 0, six of length 1, twelve of length $\sqrt{2}$, and so on. Two lattices with the same number of elements of every length are called *isospectral.*

In 1, 2, and 3 dimensions, isospectral lattices are necessarily isometric, meaning that one lattice can be reached from the other via rotations and/or reflections. But this is no longer true in higher dimensions! There are several known constructions of pairs of lattices that are isospectral but not isometric. However, we do not yet have a complete classification of such pairs in any dimension $n \geq 4$.

In this project, we will begin by developing an understanding of lattices and their theta functions. We will then explore the known constructions of pairs of isospectral lattices, with the goal of working towards a classification of 4-dimensional isospectral lattices: Can we construct new examples that haven't been found before? Can we prove that we've found them all?

Prerequisites.

- Must be comfortable with linear algebra at least at the level of Math 217.
- Programming experience would be helpful but is not required.