Geodesic flow in special linear group

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A relevant research topic in the last 60 years is the study of geodesic flow in left-invariant structures in Lie groups. A Lie group is a group with smooth multiplication; a natural example is the general linear group $GL(n)$ (the n by n matrices whose determinant is different from zero). Every Lie group has a distinguished point and a canonical map from the identity point to an arbitrary point g, namely the identity and left translation $L_g(h) = g * h$, where $*$ is the multiplication. The tangent space of the identity is a vector space equipped with a Lie bracket, making the tangent space a Lie algebra. Using the left translation, we push the Lie algebra structure at every tangent space.

A Riemanian metric on a Lie group G is a smooth scalar product $< , >$ at every tangent space. We say that a scalar product is left-invariant if the scalar product is G invariant, i.e.,

$$< v_1, v_2 >_h = < (L_g)_* v_1, (L_g)_* v_2 >_{gh} \quad \text{for all} \quad v_1 \ \text{ and } \ v_2 \in T_h G,$$

where $(L_g)_*$ is the push-forward of the left-translation, i.e., the differential of the left-translation.

A geodesic is a locally minimizing curve, i.e., the shorter curve joining two points in a neighborhood. The study of geodesic flow in left-invariant structure is reduced to the study of the Lie algebra. We will use a left-invariant structure to study the geodesic flow in special linear group $SL(2)$ (the 2 by 2 matrices whose determinant is equal to 1).