Borrowing A Reusable Resource

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Abstract

If agents have private and independent valuations, then there are efficient mechanisms that are dominant strategy incentive compatible (DSIC). This paper focuses on creating efficient mechanisms and analyzing untruthful equilibria when agents have interdependent valuations. When the manager knows agents' independent types, choosing one agent to pay the other the value they obtained from using the resource is a pricing mechanism that is ex-post incentive compatible and ex-post budget balanced. In such a mechanism, I show that every untruthful ex-post Nash equilibrium must have the same allocation decision as the truthful ex-post Nash equilibrium. When the manager does not know agents' value functions or types, I construct a two-stage mechanism that is ex-post incentive compatible and ex-ante budget balanced. Given the mechanism, I show that an untruthful ex-post Nash equilibrium exists and reporting truthfully is preferred over the untruthful equilibrium depending on agents' types and valuations.

Keywords: Sharing Economy, Interdependent Valuations, Mechanism Design.

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1 Introduction

Given the independent private valuations model, previous literature has shown that the Vickrey-Clarke-Groves (VCG) mechanism is DSIC. Efficient mechanisms become more complicated when agents' values are interdependent. This paper focuses on constructing two incentive compatible and budget balanced efficient mechanisms when agents' valuations are interdependent.

Imagine a situation in which agents need to borrow a reusable resource for some time, also known as an agent's type, and there is a manager who must make the decision of allocating the resource to the agents. Each agent has private information about the time they will need to borrow the resource and a valuation, which is a function of the time it takes for a resource to be used. When an agent obtains the resource, he may choose to use it for however long he desires, during which time other agents do not use the resource. For the agents that do not use the resource, I assume that their valuations depend on the types of agents who use the resource. Hence, I consider the case that agents' valuations are interdependent.

There are numerous examples of when agents have interdependent valuations. A classic example is the model proposed by Akerlof (1970), where a seller has private information about the quality of a good, and the buyer's valuation depends on a belief of such information conditional on the price. Another example is given by Milgrom and Weber (1982), where in a mineral-rights auction, bidders have valuations that are influenced by their private signals about the value of the tract and what they infer about others' signals. My model differs from those proposed in that the resource or good is only used for a period of time, and then the resource is returned back to the manager (i.e., the resource is never bought or sold; instead, it is borrowed or lent). An agent's valuation of the resource is thus directly influenced by when they can use the resource and the amount of time they need to use it.

Vast swaths of industries throughout our current sharing economy also involve the lending and borrowing of reusable resources, whose consumers have valuations depending on how long and when they can use the resource. Take, for example, the ride-sharing, home, and vehicle leasing industries, whose market sizes comprise of billions of dollars. Even for the government (who acts as the manager), there are instances in which some reusable resource needs to be allocated among the public. Imagine a plot of land in which companies vie to acquire. Or imagine agents attempting to enter a country but must be searched before entering. In the latter case, the resource would be the officers searching the vehicles (assuming that officers only search one vehicle at a time).

In the transportation industry, the interdependent valuation assumption also applies to many situations. Imagine two boats that must pass a canal, but only one can pass at a time[1]; or imagine two planes that need to take off at a runway, but only one may take off at a time; or imagine two spacecraft that need to land, but only one can land on the platform at a time; or imagine two trains traveling in the same direction that must use the same track for some time, but only one can use the track at a time. These examples are, again, only a tiny theme of the broader topic of agents borrowing a resource for some time.

In economics, the definition of efficiency may change depending on the context in which it is used. In this paper, I focus on *allocative efficiency*, which occurs when resources are distributed or allocated in a way such that social welfare is maximized[2]. In designing models that include interactions between the manager and the agents, we want to allocate the resource to the agent that maximizes social welfare. Therefore, I create welfare-maximizing mechanisms throughout my paper and neglect investigating mechanisms with other maximization objectives (e.g., maximizing revenue for the manager). In order to accurately calculate social welfare, I investigate mechanisms that result in agents reporting truthfully as some equilibrium. Assuming agents report truthfully, the mechanisms I consider are budget balanced.

The rest of the paper is organized as follows. Section 2 introduces previous literature on the theory of mechanism design with independent and interdependent valuations as well as applications for the models I propose. Section 3 presents a model when the manager knows agents' values and can observe agent's types (but the manager does not know agents' types before they reveal it, that is, if they do). Given such a model, section 4 proposes an ex-post incentive compatible and ex-post budget balanced mechanism. Section 5 provides a characterization of ex-post Nash equilibria and addresses the possibility of collusion, given the mechanism introduced in section 4. Section 6 introduces a model when the manager does not know agents' values and cannot observe agent's types. Given such a model, section 7 proposes a mechanism that is ex-post incentive compatible and ex-ante budget balanced. Section 8 presents a framework for analyzing untruthful ex-post Nash equilibria and conditions on agents' value functions and types such that the truthful ex-post Nash equilibrium is unique. Section 9 presents a model to analyze the changing nature of the manager's budget across several

[1]See the recent Panama Canal Auction that took place involving millions of dollars to determine which boats should be allowed to cross at some time.

[2]It can be proven by induction on the number of agents that if the welfare function is linear, then social welfare is maximized if and only if allocative efficiency is achieved. Hence, these two terms are equivalent in my analysis.

rounds, given the ex-ante budget balanced mechanism found in section 7. Section 10 analyzes how the manager's budget may change over several rounds.

2 Literature Review

Previous research by Vickrey (1961); Clarke (1971); Groves (1973) have shown that when agents have private types, valuations depend only on agents' types, and utilities are quasilinear, then the VCG mechanism is efficient and DSIC. To achieve budget balanced, d'Aspremont and Gérard-Varet (1979); Arrow (1979) have shown that an additional constant added onto the transfer rule of the VCG mechanism is necessary. Other papers like Börgers and Norman (2009) have shown modifications of the additional constant to transform a mechanism from ex-ante to ex-post budget balanced. But when valuations depend on the types of all agents (i.e., valuations are interdependent) and types are independent, previous literature by Ausubel (2004); Dasgupta and Maskin (2000) have shown that the VCG mechanism fails incentive compatibility.

However, Mezzetti (2004) shows that if, in addition to reporting types, agents are allowed to report their payoffs after the outcome decision has been made (which previous literature has neglected), truthfully reporting one's types is an ex-post Nash equilibrium. Hence, I also set up two reporting stages and analyze ex-post Nash equilibria.

The main difference between my work and his is that while Mezzetti focuses on only analyzing truthful ex-post Nash equilibrium, I examine the general structure of equilibria and consider untruthful equilibrium. As a result of the analysis of untruthful equilibrium, I then consider agents' preference over the equilibria. A more subtle contribution I make is extending Mezzetti's work by including value functions into what he defines as an agent's type. Hence, I show that his analysis is relevant even if an agent's type includes more than just a subset of the real numbers.

Further, a plethora of literature has researched the implementation and applications of mechanisms under certain models. Some of these papers discuss applications that are related to the models that I propose. One such application is in the field of intersection management (and congestion management in general), where agents have a value for crossing the intersection at some time or for the right to cross some spaces at some time. Currently, this field focuses on when agents have independent private values (IPV). For an example of models and mechanisms that include the IPV assumption, see Schepperle and Böhm (2007); Vasirani and Ossowski

(2012); Carlino et al. (2013), among many others. As a result of the IPV assumption, agents report their values (i.e., make bids) over a set of goods for the manager to determine the allocation of goods. Under the IPV assumption, submitting bids to the mechanism designer effectively turns such mechanisms into variations of the VCG mechanism. Other applications include the renting of resources by the government. For example, a government may rent out land for individuals to use (perhaps for agricultural use) throughout the year.

In all of the applications discussed, if agents' valuations are interdependent and their values and types are stochastically independent, two-stage mechanisms should be implemented to achieve ex-post incentive compatibility and ex-ante budget balanced when the goal is to maximize social welfare.

3 Model with Manager Knowing Agents' Values & Types

Imagine a reusable resource that two agents would like to borrow at some time, but only one agent may borrow the resource at a time. A manager owns this resource, and his permission is required for an agent to use it. Let $I = \{1, 2\}$ be the set containing the two agents. The right to borrow the resource can also be considered a good. Therefore, this model has two goods (which include using the resource first and second), and each agent obtains exactly one good.

I assume that each agent $i \in I$ knows $t_i \in [0, 1]$, the amount of time they expect to need to use the resource. Other agents and the manager only know this value after agent i uses the resource. Hence, each agent has a private type (or private information) that becomes commonly known to the other agent and the manager only after the agent uses the resource. The interpretation of t_i is that for any agent $i \in I$, if $t_i = 1$, then agent i expects to take the maximum amount of time (out of all agents) to use the resource. For example, suppose that the maximum time that any agent expects to use the resource is one year. If $t_1 = 1$, then agent 1 expects to use the resource for one year.

Further, for each agent $i \in I$, the value function $v_i(t)$ represents agent i's willingness to pay (WTP) for using the resource at time t. The interpretation of this value function is that $t = 0$ represents the beginning of when agents desire to use the resource, and $v_i(0)$ represents agent i's WTP when they are able to use the resource instantly[3]. I limit the domain of this function to the interval $[0, 2]$ and its range to the interval $[-1, 1]$. The

[3] Obviously, in some cases, using the resource instantly is impossible, even if the agent wanted to. However, there may be cases when using the resource immediately is possible. Hence, I include this aspect within my model for both technical reasons and to include such cases.

number 1 in the interval is the maximum amount of money agents 1 and 2 are willing to spend. Hence, numbers in the range of the value function are normalized to the maximum amount of money all agents are willing to spend to borrow the resource. For example, if the maximum amount of money that both agents are willing to spend is $20 USD to use the resource, then 1 represents $20 USD, and 0.5 represents $10 USD. Likewise, the number 1 in the domain should also be thought of as some normalized maximum time either set by the manager or by agents. Since there are two agents, the domain of the value function includes the interval $[1, 2]$.

In this section, the primary assumption that I make about the value function $v_i(t)$ for all $i \in I$, is that the manager *knows* the value function $v_i(t)$ of all agents. Formally, for all agent $i \in I$, I let $v_i \in V = \{f : [0, 2] \to [-1, 1]\}$. It may be important to note that the value functions may not be monotonically decreasing in the estimated time of using the resource, and this implies that some agents may be willing to pay more to acquire the resource at some later time. For a realistic example of this situation, think of when the government leases out farmland to farmers throughout the year. Depending on the season, the farmland may or may not be more valuable.

I now describe the utility of each agent. For all $i, j \in I$ such that $i \neq j$, agent i's utility if he is allowed to use the resource first at t_i time and receives a transfer $T_i > 0$ from (or pays a transfer $T_i < 0$ to) the manager is $v_i(t_i) + T_i$, and agent i's utility if he receives the resource second at $t_i + t_j$ time and receives a transfer $T_i > 0$ from (or pays a transfer $T_i < 0$ to) the manager is $v_i(t_i + t_j) + T_i$. The manager's utility is $\sum_{i \in I} T_i$ after she obtains the transfers from the agents.

In section 4, I focus on analyzing a subclass of mechanisms, defined by definition 1. In the following definition, Ω denotes the set of probabilities, over I agents, representing who should be allowed to use the resource first. Formally, $\Omega := \{(1, 0), (0.5, 0.5), (0, 1)\}$ where the first component of an element of Ω represents the probability that agent 1 is allowed to use the resource first, and the second component is the probability that agent 2 is allowed to use the resource first. Hence, for all $\omega \in \Omega$, $\sum_{i \in I} \omega_i = 1$, implying that one agent must be allowed to use the resource first.

Definition 1. *A direct mechanism consists of the functions q and T_i for all $i \in I$ where*

$$q : [0, 1]^2 \to \Omega$$

and

$$T_i : [0,1]^4 \to \mathbb{R}$$

The interpretation of the allocation rule q is that it determines who should use the resource first and second when agents are asked to simultaneously report their estimated times of using the resource. For each agent $i \in I$, agent i's reported type is a function of his type t_i and is denoted by $m_i \in M := \{f : [0,1] \to [0,1]\}$. Hence, the allocation rule takes as inputs $m_1(t_1)$ and $m_2(t_2)$. Similarly, for each agent $i \in I$, the mapping T_i (i.e., the transfer rule) takes the reported types $m_1(t_1)$ and $m_2(t_2)$ as well as agents' types t_1 and t_2 as inputs and outputs the payment that agent i makes or receives. If the output is positive, this implies that agent i receives such payment; if the output is negative, agent i makes such payment.

I now define two conditions the manager must satisfy when choosing a direct mechanism to decide who should be able to use the resource first.

Definition 2. *Let a direct mechanism be defined by definition 1. For each agent $i \in I$, let $m_i \in M$ be a function of agent i's type where m_i is said to be agent i's strategy or reported type (see Definition 7.D.1 in Mas-Colell et al. (1995) for details). The strategy profile $(m_1, m_2) \in M^2$ is an ex-post Nash equilibrium in the direct mechanism if for all $v_1 \in V$, $t_1, t_2 \in [0,1]$, and $m_1' \in M$,*

$$\begin{aligned} &q\left(m_1\left(t_1\right), m_2\left(t_2\right)\right) \cdot \left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right) + T_1\left(m_1\left(t_1\right), m_2\left(t_2\right), t_1, t_2\right) \\ &\geq q\left(m_1'\left(t_1\right), m_2\left(t_2\right)\right) \cdot \left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right) + T_1\left(m_1'\left(t_1\right), m_2\left(t_2\right), t_1, t_2\right) \end{aligned}$$

and for all $v_2 \in V$, $t_1, t_2 \in [0,1]$, and $m_2' \in M$,

$$\begin{aligned} &q\left(m_1\left(t_1\right), m_2\left(t_2\right)\right) \cdot \left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right) + T_2\left(m_1\left(t_1\right), m_2\left(t_2\right), t_1, t_2\right) \\ &\geq q\left(m_1\left(t_1\right), m_2'\left(t_2\right)\right) \cdot \left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right) + T_2\left(m_1\left(t_1\right), m_2'\left(t_2\right), t_1, t_2\right) \end{aligned}$$

A direct mechanism is ex-post incentive compatible if for all agents, reporting truthfully is an ex-post Nash equilibrium (i.e., the strategy profile where for all $i \in I$ and $t_i \in [0,1], m_i\left(t_i\right) = t_i$ is an ex-post Nash equilibrium).

Definition 3. *A direct mechanism q and T_i for all $i \in I$ is ex-post budget balanced if*

$$\sum_{i \in I} T_i = 0$$

The economic interpretation of definition 3 is that the direct mechanism is designed such that the manager does not earn a profit. Hence, the money the manager receives equals the money the manager pays.

4 Ex-Post Incentive Compatible & Ex-Post Budget Balanced Mechanism

This section introduces an ex-post incentive compatible and ex-post budget balanced mechanism. The efficient allocation function that I choose to analyze is as follows. To simplify notation for eqs. (4.1) and (4.2), for each agent $i \in I$, let m_i be the output of agent i's reported type.

$$q(m_1, m_2) = \begin{cases} (1,0) \text{ if } v_1(m_1) + v_2(m_1 + m_2) > v_2(m_2) + v_1(m_1 + m_2) \\ (0.5, 0.5) \text{ if } v_1(m_1) + v_2(m_1 + m_2) = v_2(m_2) + v_1(m_1 + m_2) \\ (0,1) \text{ if } v_1(m_1) + v_2(m_1 + m_2) < v_2(m_2) + v_1(m_1 + m_2) \end{cases} \tag{4.1}$$

An interpretation of the allocation function is that an agent can use the resource first if the reported social welfare is greater when he uses the resource first than when he uses the resource second. If the reported social welfare is the same when agent 1 or 2 uses the resource first, then an agent is randomly selected to use the resource first, with each agent having a 50% chance of being selected.

The transfer function that I choose to analyze is as follows.

$$\begin{aligned} T_1(m_1, m_2, t_1, t_2) &= -q(m_1, m_2) \cdot (v_1(t_1), v_1(t_1 + t_2)) \\ T_2(m_1, m_2, t_1, t_2) &= q(m_1, m_2) \cdot (v_1(t_1), v_1(t_1 + t_2)) \end{aligned} \tag{4.2}$$

In other words, agent 1 pays (or receives from) agent 2, the valuation agent 1 obtained from using the resource. Agent 2 receives (or pays) agent 1's valuation. Agent 1 makes a payment if his valuation is positive (for all possible types) and receives a payment if his valuation is negative (for all possible types). Any agent may receive or pay money because the range of v includes both negative and positive numbers.

Proposition 1. *The direct mechanism defined by eqs.* (4.1) *and* (4.2) *is ex-post incentive compatible and ex-post budget balanced.*

Proof. By adding the transfer functions described by eq. (4.2), the mechanism is ex-post budget balanced. To show that the mechanism is ex-post incentive compatible, first consider the perspective of agent 1. I will show that for all $v_1 \in V$, $t_1, t_2 \in [0,1]$, and $m_1 \in M, q(t_1,t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) + T_1(t_1,t_2,t_1,t_2) = q(m_1(t_1),t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) + T_1(m_1(t_1),t_2,t_1,t_2)$. Let arbitrary $v_1 \in V$ and $t_1, t_2 \in [0,1]$, and consider the utility of agent 1 when agent 1 arbitrarily reports $m_1 \in M$ and agent 2 truthfully reports t_2.

$$\begin{aligned}
&q(t_1,t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) + T_1(t_1,t_2,t_1,t_2) \\
&= q(t_1,t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) - q(t_1,t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) \\
&= 0 \\
&= q(m_1(t_1),t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) - q(m_1(t_1),t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) \\
&= q(m_1(t_1),t_2) \cdot (v_1(t_1), v_1(t_1+t_2)) + T_1(m_1(t_1),t_2,t_1,t_2)
\end{aligned}$$

Notice that the utility of agent 1 is always zero, no matter what his reported types are. This implies that reported types ultimately do not affect agent 1's utility. Thus, for agent 1, his utility from reporting truthfully is the same as his utility from misreporting.

Likewise, similar analysis can be done from the perspective of agent 2 in order to show that for all $v_2 \in V$, $t_1, t_2 \in [0,1]$, and $m_2 \in M$,

$$\begin{aligned}
&q(t_1,t_2) \cdot (v_2(t_1+t_2), v_2(t_2)) + T_2(t_1,t_2,t_1,t_2) \\
&\geq q(t_1,m_2(t_2)) \cdot (v_2(t_1+t_2), v_2(t_2)) + T_2(t_1,m_2(t_2),t_1,t_2).
\end{aligned}$$

Let arbitrary $v_2 \in V$ and $t_1, t_2 \in [0,1]$. Consider the utility of agent 2 when agent 1 truthfully reports his type (i.e., $m_1(t_1) = t_1$ for any $t_1 \in [0,1]$), and agent 2 reports arbitrary $m_2 \in M$.

$$
\begin{aligned}
&q\left(t_1, m_2(t_2)\right) \cdot \left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right) + T_2\left(t_1, m_2(t_2), t_1, t_2\right) \\
&= q\left(t_1, m_2(t_2)\right) \cdot \left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right) + q\left(t_1, m_2(t_2)\right) \cdot \left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right) \\
&= q\left(t_1, m_2(t_2)\right) \cdot \left(v_2\left(t_1+t_2\right) + v_1\left(t_1\right), v_1\left(t_1+t_2\right) + v_2\left(t_2\right)\right)
\end{aligned}
$$

Case 1. Suppose that both agents report truthfully, and it is in fact the case that $v_1\left(t_1\right) + v_2\left(t_1+t_2\right) > v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$, then 2's utility is $v_2\left(t_1+t_2\right) + v_1\left(t_1\right)$. If agent 2 misreports m_2 such that $v_1\left(t_1\right) + v_2\left(t_1+m_2(t_2)\right) < v_2\left(m_2(t_2)\right) + v_1\left(t_1+m_2(t_2)\right)$, then 2's utility is $v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$, which is less than the utility gained when reporting truthfully since we know that $v_1\left(t_1\right) + v_2\left(t_1+t_2\right) > v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$. If agent 2 misreports m_2 such that $v_1\left(t_1\right) + v_2\left(t_1+m_2(t_2)\right) = v_2\left(m_2(t_2)\right) + v_1\left(t_1+m_2(t_2)\right)$, then 2's utility is $0.5v_2\left(t_2\right) + 0.5v_1(t_1) + 0.5v_1\left(t_1+t_2\right) + 0.5v_2(t_1+t_2)$. But we know that

$$
v_1\left(t_1\right) + v_2\left(t_1+t_2\right) > v_2\left(t_2\right) + v_1\left(t_1+t_2\right)
$$

$$
\Leftrightarrow
$$

$$
v_2\left(t_1+t_2\right) + v_1\left(t_1\right) > 0.5v_2\left(t_2\right) + 0.5v_1\left(t_1\right) + 0.5v_1\left(t_1+t_2\right) + 0.5v_2\left(t_1+t_2\right)
$$

Case 2. Suppose that both agents report truthfully, and it is in fact the case that $v_1\left(t_1\right) + v_2\left(t_1+t_2\right) < v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$, then 2's utility is $v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$. Similar to the analysis above, agent 2's utility when misreporting will again be less than the utility gained when reporting truthfully by considering similar subcases as those considered in the case above.

Case 3. Finally, suppose that both agents report truthfully, and it is, in fact, the case that $v_1\left(t_1\right) + v_2\left(t_1+t_2\right) = v_2\left(t_2\right) + v_1\left(t_1+t_2\right)$, then 2's expected utility is

$$
\begin{aligned}
0.5v_2\left(t_2\right) + 0.5v_1\left(t_1\right) + 0.5v_1\left(t_1+t_2\right) + 0.5v_2\left(t_1+t_2\right) &= v_2\left(t_2\right) + v_1\left(t_1+t_2\right) \\
&= v_1\left(t_1\right) + v_2\left(t_1+t_2\right)
\end{aligned}
$$

If agent 2 misreports m_2 such that $v_1\left(t_1\right) + v_2\left(t_1+m_2(t_2)\right) < v_2\left(m_2(t_2)\right) + v_1\left(t_1+m_2(t_2)\right)$ or $v_1\left(t_1\right) + v_2\left(t_1+m_2(t_2)\right) > v_2\left(m_2(t_2)\right) + v_1\left(t_1+m_2(t_2)\right)$, then agent 2's utility is equal to the utility he obtains when reporting truthfully.

This proof shows that the direct mechanism is ex-post incentive compatible and budget balanced. □

Proposition 1 implies that it is in the best interest of each agent to report truthfully if other agents also report truthfully. Hence, it also implies Bayesian Incentive Compatibility. Further, if agents do indeed report truthfully, then eq. (4.2) implies that the manager will not lose or gain any money. However, because I have shown only ex-post incentive compatibility and not DSIC, it is possible that agents may collude and simultaneously misreport.

5 Characterization of Ex-Post Nash Equilibria

In this section, I analyze the general structure of ex-post Nash equilibria resulting from the direct mechanism defined by eqs. (4.1) and (4.2). I find conditions that every untruthful ex-post Nash equilibrium should satisfy, and if untruthful ex-post Nash equilibria exist, then the utility an agent gains from the truthful ex-post Nash equilibrium is no less than the utility the agent gains from any untruthful ex-post Nash equilibrium. Hence, agents do not have an incentive to deviate from reporting truthfully.

Proposition 2. *Let the direct mechanism defined by eqs.* (4.1) *and* (4.2). *The strategy profile* (m_1, m_2) *is an ex-post Nash equilibrium if and only if for all* $v_1, v_2 \in V$ *and* $t_1, t_2 \in [0,1]$

1. $q(m_1(t_1), m_2(t_2)) = (1,0)$ *whenever* $v_1(t_1) + v_2(t_1+t_2) > v_2(t_2) + v_1(t_1+t_2)$ *and*
2. $q(m_1(t_1), m_2(t_2)) = (0,1)$ *whenever* $v_1(t_1) + v_2(t_1+t_2) < v_2(t_2) + v_1(t_1+t_2)$ *and*
3. $q(m_1(t_1), m_2(t_2)) = (0.5,0.5)$ *whenever* $v_1(t_1) + v_2(t_1+t_2) = v_2(t_2) + v_1(t_1+t_2)$

Proof. Consider agent 2, since, again, agent 1 is indifferent between any strategy profile shown in proposition 1.

($\Leftarrow$) Suppose for all $v_1, v_2 \in V$ and $t_1, t_2 \in [0,1]$, the three conditions hold. In order for (m_1, m_2) to be an ex-post Nash equilibrium, we must have that for all $m_2' \in M$

$$\begin{aligned} &q(m_1(t_1), m_2(t_2)) \cdot (v_2(t_1+t_2) + v_1(t_1), v_2(t_2) + v_1(t_1+t_2)) \\ &\geq q(m_1(t_1), m_2'(t_2)) \cdot (v_2(t_1+t_2) + v_1(t_1), v_2(t_2) + v_1(t_1+t_2)) \end{aligned}$$

WLOG, suppose $v_1(t_1) + v_2(t_1+t_2) > v_2(t_2) + v_1(t_1+t_2)$, then we immediately have $q(m_1(t_1), m_2(t_2)) =$

$(1,0)$, and hence for any $m_2' \in M$

$$v_2(t_1+t_2)+v_1(t_1) \geq q(m_1(t_1), m_2'(t_2)) \cdot (v_2(t_1+t_2)+v_1(t_1), v_2(t_2)+v_1(t_1+t_2))$$

The other two cases are similar.

($\Rightarrow$) Suppose the strategy profile (m_1, m_2) is an ex-post Nash equilibrium. Let arbitrary $v_1, v_2 \in V$ and $t_1, t_2 \in [0,1]$.

Case 1. Suppose $v_1(t_1)+v_2(t_1+t_2) > v_2(t_2)+v_1(t_1+t_2)$. For the sake of contradiction, suppose $q(m_1(t_1), m_2(t_2)) = (0.5, 0.5)$ or $q(m_1(t_1), m_2(t_2)) = (0,1)$.

Consider the case when $q(m_1(t_1), m_2(t_2)) = (0.5, 0.5)$. Then we would need

$$0.5(v_2(t_1+t_2)+v_1(t_1)+v_2(t_2)+v_1(t_1+t_2)) \geq v_2(t_1+t_2)+v_1(t_1)$$

$$\Leftrightarrow$$

$$v_2(t_2)+v_1(t_1+t_2) \geq v_2(t_1+t_2)+v_1(t_1)$$

But obviously $v_2(t_2)+v_1(t_1+t_2) \geq v_2(t_1+t_2)+v_1(t_1)$ is not possible since $v_1(t_1)+v_2(t_1+t_2) > v_2(t_2)+v_1(t_1+t_2)$.

Next, consider when $q(m_1(t_1), m_2(t_2)) = (0,1)$. Then we would need

$$v_2(t_2)+v_1(t_1+t_2) \geq 0.5(v_2(t_1+t_2)+v_1(t_1)+v_2(t_2)+v_1(t_1+t_2))$$

$$\Leftrightarrow$$

$$v_2(t_2)+v_1(t_1+t_2) \geq v_2(t_1+t_2)+v_1(t_1)$$

But again this is not possible since $v_1(t_1)+v_2(t_1+t_2) > v_2(t_2)+v_1(t_1+t_2)$. Hence, m_1 and m_2 must be selected such that $q(m_1(t_1), m_2(t_2)) = (1,0)$.

Case 2. Suppose $v_1(t_1)+v_2(t_1+t_2) < v_2(t_2)+v_1(t_1+t_2)$. For the sake of contradiction again, suppose $q(m_1(t_1), m_2(t_2)) = (0.5, 0.5)$ or $q(m_1(t_1), m_2(t_2)) = (1,0)$.

If $q(m_1(t_1), m_2(t_2)) = (0.5, 0.5)$, then we would need

$$0.5(v_2(t_1+t_2) + v_1(t_1) + v_2(t_2) + v_1(t_1+t_2)) \geq v_2(t_2) + v_1(t_1+t_2)$$

$$\Leftrightarrow$$

$$v_2(t_1+t_2) + v_1(t_1) \geq v_2(t_2) + v_1(t_1+t_2)$$

But obviously $v_2(t_1+t_2) + v_1(t_1) \geq v_2(t_2) + v_1(t_1+t_2)$ is not possible since $v_1(t_1) + v_2(t_1+t_2) < v_2(t_2) + v_1(t_1+t_2)$.

If $q(m_1(t_1), m_2(t_2)) = (1, 0)$, then we would need

$$v_2(t_1+t_2) + v_1(t_1) \geq v_2(t_2) + v_1(t_1+t_2)$$

But again this is not possible since $v_1(t_1) + v_2(t_1+t_2) > v_2(t_2) + v_1(t_1+t_2)$. Hence, m_1 and m_2 must be selected such that $q(m_1(t_1), m_2(t_2)) = (0, 1)$.

Case 3. Lastly, note that if $v_1(t_1) + v_2(t_1+t_2) = v_2(t_2) + v_1(t_1+t_2)$, then we automatically have for any outcome of q,

$$\begin{aligned} & q(m_1(t_1), m_2(t_2)) \cdot (v_2(t_1+t_2) + v_1(t_1), v_2(t_1+t_2) + v_1(t_1)) \\ & = q(m_1(t_1), m_2'(t_2)) \cdot (v_2(t_1+t_2) + v_1(t_1), v_2(t_1+t_2) + v_1(t_1)). \end{aligned}$$

This shows that if a strategy profile is an ex-post Nash equilibrium, then it will satisfy the following three properties. More importantly, an ex-post Nash equilibrium is mathematically equivalent to the three listed conditions.

□

The reverse implication ($\Leftarrow$) can be interpreted as follows. If a supposedly untruthful strategy profile can achieve the same outcome decisions as the truthful symmetric strategy profile for all possible values and types, then the untruthful strategy profile is also an ex-post Nash equilibrium. In short, this implication is an easy way to check whether a strategy profile is an ex-post Nash equilibrium. On the contrary, the forward implication ($\Rightarrow$) helps to check if a strategy profile is not an ex-post Nash equilibrium. Hence, it will be used in the following corollary.

In analyzing untruthful strategic profiles, we may choose to consider either symmetric (i.e., $m_1(t) = m_2(t)$

for all $t \in [0, 1]$) or asymmetric (i.e., not symmetric) strategies. For example, suppose in the model described in section 3, types and value functions are not private information (i.e., agents know each other's types and value functions). When this is the case, it can be shown that many untruthful asymmetric strategy profiles exist that are an ex-post Nash equilibrium.

On the other hand, untruthful symmetric strategy profiles are interesting because agents do not need to know much information about the other agents or the manager to choose their best response. Hence, I proceed by considering a form of symmetric strategy profiles that cannot be ex-post Nash equilibria.

Corollary 1. *Both agents reporting a constant $k \in [0, 1]$ is not an ex-post Nash equilibrium in the direct mechanism defined by eqs.* (4.1) *and* (4.2)*.*

Proof. This is a result of proposition 2. Suppose, by contradiction, that the proposed strategy profile is an ex-post Nash equilibrium. Reporting a constant will cause the allocation function to equal $(1, 0)$, $(0.5, 0.5)$, or $(0, 1)$. In either of these three cases, the strategy profile will contradict the results of proposition 2. This can also be shown using the contrapositive of proposition 2. □

Corollary 1 shows that unlike the dominant strategy of reporting 0 that arises from a VCG mechanism under the interdependent values model, reporting 0 or any other constant as the symmetric strategy profile is not an ex-post Nash equilibrium in the direct mechanism defined by eqs. (4.1) and (4.2).

It is important to note that even if untruthful ex-post Nash equilibria exist and agents reach these other untruthful ex-post Nash equilibria, their utilities in such equilibria are no greater than the utilities from reporting truthfully.

Corollary 2. *For all untruthful ex-post Nash equilibria and all agents $i \in I$, the utility agent i gains from the truthful ex-post Nash equilibrium is no less than the utility gained from the untruthful ex-post Nash equilibrium.*

Proof. This almost directly follows from proposition 1. For each agent, instead of considering when the other agent reports truthfully, consider when they do not report truthfully. The utility gained from misreporting will always be equal or less than the utility gained from truthfully reporting. □

Corollary 2 shows that when the manager knows agents' types and values, he is able to prevent agents from obtaining a higher payoff when they misreport than when they report truthfully.

6 Model with Manager not Knowing Agents' Values & Types

In this section, I will include features of the model described in section 3, but now I consider the case when the manager never observes agents' times or knows agents' value functions. As a result, the manager sets up two reporting stages. In the first stage, the manager still asks each agent to report an estimated time of using the resource (similar to section 3) and a value function. Using these reports, the manager makes an allocation decision. After allocations have been made, in the second stage, the manager asks agents to report their outcome payoffs. Transfers are then determined based on second-stage reports.

To formally describe the model, there is still a reusable resource that two agents would like to borrow. The agents are represented by 1 and 2 in the set I. I assume that each agent $i \in I$ has private information $t_i \in [0,1]$ before and when allocation decisions are made. Hence, other agents and the manager do not know this value before the agent borrows the resource. However, the manager has a belief of t_i. Formally, for any $i \in I$, let the tuple $([0,1], \mathcal{B}([0,1]), \mu_i)$ be a probability space where $\mathcal{B}(X)$ is defined to be the Borel σ-algebra on X and $\mu_i : \mathcal{B}([0,1]) \to [0,1]$ is the probability measure; μ_i is also known as the marginal probability distribution of t_i. Let the product probability space be $\left([0,1]^2, \mathcal{B}\left([0,1]^2\right), \mu\right)$ where μ is the product probability measure (i.e., $\forall x, y \in \mathcal{B}([0,1]), \mu(x \times y) = \mu_1(x)\mu_2(y)$ and μ is a nonnegative countably additive set function such that $\mu\left([0,1]^2\right) = 1$). The manager knows the probability distribution induced by the probability measure μ.

Further, another addition that I add in this section is that for all $i \in I$, agent i's value function $v_i \in V$ is private information where V is a finite set containing mappings from $[0,2]$ to $[-1,1]$. Since the manager has a belief of v_i, we would like to define a probability measure on V and, more generally, V^2. Hence, for all $i \in I$, let η_i be the probability measure defined on $\mathcal{P}(V)$, the power set of V, such that for any $x \in V, \eta_i(\{x\})$ represents the probability that agent i's value function is x. Note that since η_i is a probability measure, it satisfies $\eta_i(V) = 1$, and if $x_1, \ldots, x_n \in \mathcal{P}(V)$ is a countable sequence of disjoint sets, then $\eta_i\left(\cup_{j=1}^n x_j\right) = \sum_{j=1}^n \eta_i(x_j)$. The latter fact implies that given $\eta_i(\{x\})$ for all $x \in V$, it is possible to define $\eta_i\left(\cup_{j=1}^n x_j\right)$ given a countable sequence of disjoint sets $x_1, \ldots, x_n$. Finally, let η be the product probability measure defined on $\mathcal{P}\left(V^2\right)$. The manager knows the probability distribution induced by the probability measure η.

The utilities of the agents remain the same. For all $i, j \in I$ such that $i \neq j$, agent i's utility if he uses the resource first at t_i time and receives a transfer $T_i > 0$ from (or pays a transfer $T_i < 0$ to) the manager is $v_i(t_i) + T_i$, and agent i's utility if he uses the resource second at $t_i + t_j$ time and receives a transfer $T_i > 0$ from (or pays a transfer $T_i < 0$ to) the manager is $v_i(t_i + t_j) + T_i$. The manager's utility is $\sum_{i \in I} T_i$ after she obtains transfers from the agents.

In the following analysis, $E[g(\cdot)]$ is defined to be the expectation with respect to the probability measures μ and η. Further, $\Omega := \{(1,0), (0.5, 0.5), (0,1)\}$ again denotes a set of probabilities over I agents representing who should be allowed to use the resource first.

Definition 4. *A direct mechanism consists of the functions q and T_i for all $i \in I$ where*

$$q : [0,1]^2 \times V^2 \to \Omega$$

and

$$T_i : [0,1]^2 \times V^2 \times [-1,1]^2 \to \mathbb{R}$$

The interpretation of the allocation function q is that, in the first stage, agents simultaneously report their estimated times of using the resource and their valuation functions for the manager to determine who should be allowed to use the resource first. Similar to section 3, for each agent $i \in I$, agent i's reported type $m_i^r \in M := \{f : [0,1] \to [0,1]\}$ is a function of his type t_i. Agent i's reported value function $v_i^r \in V$ is a function of the total time it takes for agent i to obtain and use the resource. In stage one, m_i^r and v_i^r can be considered agent i's strategy. The interpretation of the transfer functions T_i is that the manager pays each agent some amount of money determined by agents' stage one reports and agents' stage two reported utility payoffs, (u_1^r, u_2^r), where these reports can be considered as agents' stage two strategies. Again, agent's strategies may only depend on the information they know (i.e., their own types) and not the types of others.

Definition 5. *A direct mechanism q and T_i for all $i \in I$ is ex-ante budget balanced if*

$$E\left[\sum_{i \in I} T_i\right] = 0.$$

Definition 6. *Let a direct mechanism be defined by definition 4. The strategy profile $(m_1^r, m_2^r, v_1^r, v_2^r, u_1, u_2) \in M^2 \times V^2 \times [-1,1]^2$ is an ex-post Nash equilibrium if for all $v_1, v_2 \in V, t_1, t_2 \in [0,1]$, and $(m_1', v_1', u_1^r) \in M \times V \times$*

$[-1,1]$

$$q\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r\right)\cdot\left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right)+T_1\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r, u_1, u_2\right)$$
$$\geq q\left(m_1'(t_1), m_2^r(t_2), v_1', v_2^r\right)\cdot\left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right)+T_1\left(m_1'(t_1), m_2^r(t_2), v_1', v_2^r, u_1^r, u_2\right)$$

and for all $v_1, v_2 \in V, t_1, t_2 \in [0,1]$, *and* $(m_2', v_2', u_2^r) \in M \times V \times [-1,1]$

$$q\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r\right)\cdot\left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right)+T_2\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r, u_1, u_2\right)$$
$$\geq q\left(m_1^r(t_1), m_2'(t_2), v_1^r, v_2'\right)\cdot\left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right)+T_2\left(m_1^r(t_1), m_2'(t_2), v_1^r, v_2', u_1, u_2^r\right).$$

A direct mechanism is ex-post incentive compatible if, for all agents, reporting truthfully is an ex-post Nash equilibrium. Specifically, first stage reports are truthful: for any agent $i \in I$ *and* $t_i \in [0,1]$, $m_i^r(t_i) = t_i$, *and for any agent* $i \in I$ *and* $t \in [0,1]$, $v_i^r(t) = v_i(t)$; *and second stage reports are also truthful:*

$$u_1\left(t_1, t_2, v_1, v_2\right)=\begin{cases} v_1\left(t_1\right) & \text{if } q\left(t_1, t_2, v_1, v_2\right)=(1,0) \\ v_1\left(t_1+t_2\right) & \text{if } q\left(t_1, t_2, v_1, v_2\right)=(0,1) \end{cases}$$

and

$$u_2\left(t_1, t_2, v_1, v_2\right)=\begin{cases} v_2\left(t_1+t_2\right) & \text{if } q\left(t_1, t_2, v_1, v_2\right)=(1,0) \\ v_2\left(t_2\right) & \text{if } q\left(t_1, t_2, v_1, v_2\right)=(0,1) \end{cases}.$$

Definition 7. *Let a direct mechanism be defined by* q *and* T_i *for all* $i \in I$. *Let* $\boldsymbol{e_1} = (m_1, m_2, \omega_1, \omega_2, w_1, w_2)$ *and* $\boldsymbol{e}_2 = (m_1', m_2', \omega_1', \omega_2', w_1', w_2')$ *be two ex-post Nash equilibria.* $\boldsymbol{e}_1$ *is strictly preferred to* $\boldsymbol{e}_2$ *if for all* $v_1, v_2 \in V, t_1, t_2 \in [0,1]$,

$$q\left(m_1(t_1), m_2(t_2), \omega_1, \omega_2\right)\cdot\left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right)+T_1\left(m_1(t_1), m_2(t_2), \omega_1, \omega_2, w_1, w_2\right)$$
$$> q\left(m_1'(t_1), m_2'(t_2), \omega_1', \omega_2'\right)\cdot\left(v_1\left(t_1\right), v_1\left(t_1+t_2\right)\right)+T_1\left(m_1'(t_1), m_2'(t_2), \omega_1', \omega_2', w_1', w_2'\right)$$

and

$$q\left(m_1(t_1), m_2(t_2), \omega_1, \omega_2\right)\cdot\left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right)+T_2\left(m_1(t_1), m_2(t_2), \omega_1, \omega_2, w_1, w_2\right)$$
$$> q\left(m_1'(t_1), m_2'(t_2), \omega_1', \omega_2'\right)\cdot\left(v_2\left(t_1+t_2\right), v_2\left(t_2\right)\right)+T_2\left(m_1'(t_1), m_2'(t_2), \omega_1', \omega_2', w_1', w_2'\right)$$

7 Ex-Post Incentive Compatible & Ex-Ante Budget Balanced Mechanism

I will introduce an ex-post incentive compatible mechanism in this section. The efficient allocation function, adapted for the given model, is as follows.

$$
\begin{aligned}
&q\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r\right) \\
&= \begin{cases} (1,0) \text{ if } v_1^r\left(m_1^r(t_1)\right) + v_2^r\left(m_1^r(t_1) + m_2^r(t_2)\right) > v_2^r\left(m_2^r(t_2)\right) + v_1^r\left(m_1^r(t_1) + m_2^r(t_2)\right) \\ (0.5, 0.5) \text{ if } v_1^r\left(m_1^r(t_1)\right) + v_2^r\left(m_1^r(t_1) + m_2^r(t_2)\right) = v_2^r\left(m_2^r(t_2)\right) + v_1^r\left(m_1^r(t_1) + m_2^r(t_2)\right) \\ (0,1) \text{ if } v_1^r\left(m_1^r(t_1)\right) + v_2^r\left(m_1^r(t_1) + m_2^r(t_2)\right) < v_2^r\left(m_2^r(t_2)\right) + v_1^r\left(m_1^r(t_1) + m_2^r(t_2)\right) \end{cases}
\end{aligned} \tag{7.1}
$$

The allocation function differs from that defined by eq. (4.1) since it involves agents' reported value function v_i^r. In section 3, I assumed that the value functions were *known*, and therefore, the manager did not ask for this information. In this section, the manager uses agents' reported types (also required in eq. (4.1)) and agents' reported value functions to make an allocation decision.

The transfer function that I analyze in this section is as follows.

$$
\begin{aligned}
T_1\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r, u_2^r\right) &= u_2^r + h \\
T_2\left(m_1^r(t_1), m_2^r(t_2), v_1^r, v_2^r, u_1^r\right) &= u_1^r + h
\end{aligned} \tag{7.2}
$$

The transfer function requires each agent to report their outcome payoff from using the resource, and it does not require knowing the actual time each agent took to use the resource (unlike eq. (4.2)). Each agent pays or receives a transfer equal to the other agent's reported outcome payoff and receives or pays some constant h. First stage reports are considered inputs in the transfer function because they indirectly contribute to each agent's transfer by determining agents' allocations, which then determine their payoffs.

Using the Lebesgue integral, the specific constant h that I consider is as follows.

$$
h := -\frac{1}{2}\left(\sum_{(v_1,v_2)\in V^2} \eta\left(v_1, v_2\right) \int_{[0,1]^2} \left(v_1\left(t_1\right) + v_2\left(t_1 + t_2\right)\right) 1_A + \left(v_2\left(t_2\right) + v_1\left(t_1 + t_2\right)\right) 1_B d\mu\right) \tag{7.3}
$$

where

$$A := \left\{ (t_1, t_2) \in [0,1]^2 \mid v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2) \right\} \tag{7.4}$$

and

$$B := \left\{ (t_1, t_2) \in [0,1]^2 \mid v_1(t_1) + v_2(t_1 + t_2) < v_2(t_2) + v_1(t_1 + t_2) \right\}. \tag{7.5}$$

Proposition 3. *The direct mechanism defined by eqs.* (7.1) *and* (7.2) *is ex-ante budget balanced and ex-post incentive compatible.*

Proof. I will first show that the mechanism is ex-ante budget balanced. Suppose all agents report truthfully both in the first and second stages. Formally, in the first stage, we have for all $i \in I$ and $t \in [0,1]$, $m_i^r(t) = t$, and for all $i \in I$ and $t \in [0,2]$, $v_i^r(t) = v_i(t)$. In the second stage, we have for all $i, j \in I$ such that $i \neq j$,

$$u_i^r = \begin{cases} v_i(t_i) & \text{if } v_i(t_i) + v_j(t_i + t_j) > v_j(t_j) + v_i(t_i + t_j) \\ v_i(t_i + t_j) & \text{if } v_i(t_i) + v_j(t_i + t_j) < v_j(t_j) + v_i(t_i + t_j) \end{cases}$$

Hence,

$$\begin{aligned} &E\left[T_1(t_1, t_2, v_1, v_2, u_2^r) + T_2(t_1, t_2, v_1, v_2, u_1^r)\right] \\ &= E\left[u_2^r + u_1^r\right] + 2h \\ &= \left(\sum_{(v_1, v_2) \in V^2} \eta(v_1, v_2) \int_{[0,1]^2} (v_1(t_1) + v_2(t_1 + t_2)) 1_A + (v_2(t_2) + v_1(t_1 + t_2)) 1_B d\mu \right) \\ &- \left(\sum_{(v_1, v_2) \in V^2} \eta(v_1, v_2) \int_{[0,1]^2} (v_1(t_1) + v_2(t_1 + t_2)) 1_A + (v_2(t_2) + v_1(t_1 + t_2)) 1_B d\mu \right) \\ &= 0 \end{aligned}$$

To show incentive compatibility, WLOG, consider agent 2. Note that in the second stage, agent 2 reporting truthfully or misreporting has no effect on his utility (since his transfer depends on agent 1's reported utility payoff). Hence, consider only agent 2's first stage reports. I will show that agent 2 maximizes his utility by truthfully reporting his type and value function when agent 1 truthfully reports his stage one and stage two reports. If agent 1 reports truthfully in both stages, then, in stage one, agent 1's strategy is $m_1^r(t) = t$ for all $t \in [0,1]$ and $v_1^r(t) = v_1(t)$ for all $t \in [0,2]$, and, in stage two, agent 1 reports $v_1(t_1)$ if he is allowed to use the resource first and $v_1(t_1 + t_2)$ if he is allowed to use the resource second. I will show that for all $v_1, v_2 \in V, t_1, t_2 \in$

$[0,1]$, and $(m_2^r, v_2') \in M \times V$,

$$q(t_1, t_2, v_1, v_2) \cdot (v_2(t_1 + t_2), v_2(t_2)) + T_2(t_1, t_2, v_1, v_2, u_1^r, u_2^r)$$
$$\geq q(t_1, m_2^r(t_2), v_1, v_2') \cdot (v_2(t_1 + t_2), v_2(t_2)) + T_2(t_1, m_2^r(t_2), v_1, v_2', u_1^r, u_2^r)$$
$$\Leftrightarrow$$
$$q(t_1, t_2, v_1, v_2) \cdot (v_1(t_1) + v_2(t_1 + t_2) + h, v_2(t_2) + v_1(t_1 + t_2) + h)$$
$$\geq q(t_1, m_2^r(t_2), v_1, v_2') \cdot (v_1(t_1) + v_2(t_1 + t_2) + h, v_2(t_2) + v_1(t_1 + t_2) + h)$$

Let arbitrary $v_1, v_2 \in V, t_1, t_2 \in [0,1]$, and consider the utility of agent 2 when agent 2 arbitrarily reports $(m_2^r, v_2') \in M \times V$ in stage one, and agent 1 truthfully reports in stage one and two.

$$q(t_1, m_2^r(t_2), v_1, v_2') \cdot (v_1(t_1) + v_2(t_1 + t_2) + h, v_2(t_2) + v_1(t_1 + t_2) + h)$$

Suppose that both agents report truthfully, then agent 2's utility is $v_1(t_1) + v_2(t_1 + t_2) + h$ if and only if $v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2)$, $0.5(v_1(t_1) + v_2(t_1 + t_2) + v_2(t_2) + v_1(t_1 + t_2) + 2h)$ if and only if $v_1(t_1) + v_2(t_1 + t_2) = v_2(t_2) + v_1(t_1 + t_2)$, and $v_2(t_2) + v_1(t_1 + t_2) + h$ if and only if $v_1(t_1) + v_2(t_1 + t_2) < v_2(t_2) + v_1(t_1 + t_2)$.

Case 1. Suppose it is the case that $v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2)$. This implies that 2's utility is $v_2(t_1 + t_2) + v_1(t_1) + h$. The interpretation is that agent 2 should use the resource second and should obtain agent 1's valuation when both agents report truthfully. Consider the subcases when agent 2's strategy is to misreport his type and value function, (m_2^r, v_2').

Subcase (i): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1(t_1) + v_2'(t_1 + m_2^r) < v_2'(m_2^r) + v_1(t_1 + m_2^r)$, and 2's utility is $v_2(t_2) + v_1(t_1 + t_2) + h$. We know that this utility is less than the utility gained from reporting truthfully since $v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2)$

Subcase (ii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1(t_1) + v_2'(t_1 + m_2^r) = v_2'(m_2^r) + v_1(t_1 + m_2^r)$, and 2's utility is $0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)\right)$. We know that

$$v_1\left(t_1\right)+v_2\left(t_1+t_2\right)>v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+h>0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$$

Subcase (iii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1\left(t_1\right)+v_2'\left(t_1+m_2^r\right)>v_2'\left(m_2^r\right)+v_1\left(t_1+m_2^r\right)$, and 2's utility is $v_2\left(t_1+t_2\right)+v_1\left(t_1\right)+h$. However, this is equal to the utility he obtains when he reports truthfully.

Case 2. Suppose it is the case that $v_1\left(t_1\right)+v_2\left(t_1+t_2\right)=v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$. This implies that 2's utility is $0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$. Consider the subcases when agent 2's strategy is to misreport his type and value function, (m_2^r, v_2').

Subcase (i): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1\left(t_1\right)+v_2'\left(t_1+m_2^r\right)<v_2'\left(m_2^r\right)+v_1\left(t_1+m_2^r\right)$, and 2's utility is $v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+h$. We know

$$v_1\left(t_1\right)+v_2\left(t_1+t_2\right)=v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)=v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+h$$

Subcase (ii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1\left(t_1\right)+v_2'\left(t_1+m_2^r\right)=v_2'\left(m_2^r\right)+v_1\left(t_1+m_2^r\right)$, and 2's utility is $0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$. However, this is equal to the utility he obtains when he reports truthfully.

Subcase (iii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1\left(t_1\right)+v_2'\left(t_1+m_2^r\right)>v_2'\left(m_2^r\right)+v_1\left(t_1+m_2^r\right)$, and 2's utility is $v_2\left(t_1+t_2\right)+v_1\left(t_1\right)+h$. But we know that

$$v_1\left(t_1\right)+v_2\left(t_1+t_2\right)=v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)=v_2\left(t_1+t_2\right)+v_1\left(t_1\right)+h$$

Case 3. Suppose it is the case that $v_1\left(t_1\right)+v_2\left(t_1+t_2\right)<v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$. This implies that 2's utility

is $v_2(t_2) + v_1(t_1+t_2) + h$. Consider the subcases when agent 2's strategy is to misreport his type and value function, (m_2^r, v_2').

Subcase (i): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1(t_1) + v_2'(t_1+m_2^r) < v_2'(m_2^r) + v_1(t_1+m_2^r)$, and 2's utility is $v_2(t_2) + v_1(t_1+t_2) + h$. However, this is equal to the utility he obtains when he reports truthfully.

Subcase (ii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1(t_1) + v_2'(t_1+m_2^r) = v_2'(m_2^r) + v_1(t_1+m_2^r)$, and 2's utility is $0.5(v_1(t_1) + v_2(t_1+t_2) + v_2(t_2) + v_1(t_1+t_2) + 2h)$. But we know

$$v_2(t_2) + v_1(t_1+t_2) > v_1(t_1) + v_2(t_1+t_2)$$

$$\Leftrightarrow$$

$$v_2(t_2) + v_1(t_1+t_2) + h > 0.5(v_1(t_1) + v_2(t_1+t_2) + v_2(t_2) + v_1(t_1+t_2) + 2h)$$

Subcase (iii): Consider when agent 2 misreports (m_2^r, v_2') such that $v_1(t_1) + v_2'(t_1+m_2^r) > v_2'(m_2^r) + v_1(t_1+m_2^r)$, and 2's utility is $v_2(t_1+t_2) + v_1(t_1) + h$. We know that this utility is less than the utility gained from reporting truthfully since $v_1(t_1) + v_2(t_1+t_2) < v_2(t_2) + v_1(t_1+t_2)$.

These cases show that truthfully reporting is an ex-post Nash equilibrium. □

8 Alternative Ex-Post Nash Equilibria & Collusion

In this section, I propose a non-truthful ex-post Nash equilibrium resulting from the direct mechanism defined by eqs. (7.1) and (7.2). I give conditions of agents' times and value functions, which, if true, implies that agents prefer one equilibrium over the other.

Proposition 4. *Let the direct mechanism q and T_i for all $i \in I$ be defined by eqs. (7.1) and (7.2). Agents reporting truthfully in the first stage and reporting (u_1, u_2) in the second stage such that*

$$u_2(t_1, t_2, v_1, v_2) = \begin{cases} -v_2(t_2) & \text{if } q(t_1, t_2, v_1, v_2) = (1, 0) \\ -v_2(t_1+t_2) & \text{if } q(t_1, t_2, v_1, v_2) = (0, 1) \end{cases}$$

and

$$u_1(t_1,t_2,v_1,v_2) = \begin{cases} -v_1(t_1+t_2) & \textit{if } q(t_1,t_2,v_1,v_2) = (1,0) \\ -v_1(t_1) & \textit{if } q(t_1,t_2,v_1,v_2) = (0,1) \end{cases}$$

is an ex-post Nash equilibrium in the direct mechanism.

Proof. In the truthful ex-post Nash equilibrium, shown in proposition 3, the non-constant part of agents' transfers ensures that agents' utilities are equal to the maximum possible value of social welfare for each realization of agents' types and value functions. This ensures that agents have no incentive to deviate from reporting truthfully when the other agent also reports truthfully.

In this equilibrium, in the second stage, an agent reports the opportunity cost imposed on him if he could hypothetically swap positions with the other agent (i.e., if an agent is allowed to obtain the resource first, swapping positions with the other agent means that the agent instead obtains the resource second). When each agent pays the opportunity cost that could have been obtained had they received the other agent's allocation decision, this opportunity cost aligns agents' utilities with the net benefit of an allocation decision.

To begin the proof, WLOG, consider agent 1. I will show for all $v_1, v_2 \in V, t_1, t_2 \in [0,1]$, and $(m_1', v_1', u_1^r) \in M \times V \times [-1,1]$

$$\begin{aligned} & q(t_1,t_2,v_1,v_2)\cdot(v_1(t_1), v_1(t_1+t_2)) + T_1(t_1,t_2,v_1,v_2,u_1^r,u_2) \\ & \geq q(m_1'(t_1),t_2,v_1',v_2)\cdot(v_1(t_1), v_1(t_1+t_2)) + T_1(m_1'(t_1),t_2,v_1',v_2,u_1^r,u_2) \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} & q(t_1,t_2,v_1,v_2)\cdot(v_1(t_1) - v_2(t_2) + h, v_1(t_1+t_2) - v_2(t_1+t_2) + h) \\ & \geq q(m_1'(t_1),t_2,v_1',v_2)\cdot(v_1(t_1) - v_2(t_2) + h, v_1(t_1+t_2) - v_2(t_1+t_2) + h) \end{aligned}$$

Let arbitrary $v_1, v_2 \in V, t_1, t_2 \in [0,1]$, and $(m_1', v_1', u_1^r) \in M \times V \times [-1,1]$.

Case 1. $q(t_1,t_2,v_1,v_2) = (1,0) \Leftrightarrow v_1(t_1) + v_2(t_1+t_2) > v_2(t_2) + v_1(t_1+t_2)$. Agent 1's utility is $v_1(t_1) - v_2(t_2) + h$.

Subcase (i): If $q(m_1'(t_1),t_2,v_1',v_2) = (0.5,0.5)$, then agent 1's utility is $0.5(v_1(t_1) - v_2(t_2) + v_1(t_1+t_2) - v_2(t_1+t_2) + 2h)$.

But we know

$$v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2)$$

$$\Leftrightarrow$$

$$v_1(t_1) - v_2(t_2) + h > 0.5(v_1(t_1) - v_2(t_2) + v_1(t_1 + t_2) - v_2(t_1 + t_2) + 2h)$$

Subcase (ii): If $q(m_1'(t_1), t_2, v_1', v_2) = (0, 1)$, then agent 1's utility is $v_1(t_1 + t_2) - v_2(t_1 + t_2) + h$. But we know

$$v_1(t_1) + v_2(t_1 + t_2) > v_2(t_2) + v_1(t_1 + t_2)$$

$$\Leftrightarrow$$

$$v_1(t_1) - v_2(t_2) + h > v_1(t_1 + t_2) - v_2(t_1 + t_2) + h$$

Case 2. $q(t_1, t_2, v_1, v_2) = (0.5, 0.5) \Leftrightarrow v_1(t_1) + v_2(t_1 + t_2) = v_2(t_2) + v_1(t_1 + t_2)$. Agent 1's utility is $0.5(v_1(t_1) - v_2(t_2) + v_1(t_1 + t_2) - v_2(t_1 + t_2) + 2h)$.

Subcase (i): If $q(m_1'(t_1), t_2, v_1', v_2) = (1, 0)$, then agent 1's utility is $v_1(t_1) - v_2(t_2) + h$. But we know

$$v_1(t_1) + v_2(t_1 + t_2) = v_2(t_2) + v_1(t_1 + t_2)$$

$$\Leftrightarrow$$

$$0.5(v_1(t_1) - v_2(t_2) + v_1(t_1 + t_2) - v_2(t_1 + t_2) + 2h) = v_1(t_1) - v_2(t_2) + h$$

Subcase (ii): If $q(m_1'(t_1), t_2, v_1', v_2) = (0, 1)$, then agent 1's utility is $v_1(t_1 + t_2) - v_2(t_1 + t_2) + h$. But we know

$$v_1(t_1) + v_2(t_1 + t_2) = v_2(t_2) + v_1(t_1 + t_2)$$

$$\Leftrightarrow$$

$$0.5(v_1(t_1) - v_2(t_2) + v_1(t_1 + t_2) - v_2(t_1 + t_2) + 2h) = v_1(t_1 + t_2) - v_2(t_1 + t_2) + h$$

Case 3. $q(t_1, t_2, v_1, v_2) = (0, 1) \Leftrightarrow v_1(t_1) + v_2(t_1 + t_2) < v_2(t_2) + v_1(t_1 + t_2)$. Agent 1's utility is $v_1(t_1 + t_2) - v_2(t_1 + t_2) + h$.

Subcase (i): If $q\left(m_1'(t_1), t_2, v_1', v_2\right) = (1, 0)$, then agent 1's utility is $v_1\left(t_1\right) - v_2\left(t_2\right) + h$. But we know

$$v_1\left(t_1\right) + v_2\left(t_1 + t_2\right) < v_2\left(t_2\right) + v_1\left(t_1 + t_2\right)$$

$$\Leftrightarrow$$

$$v_1\left(t_1 + t_2\right) - v_2\left(t_1 + t_2\right) + h > v_1\left(t_1\right) - v_2\left(t_2\right) + h$$

Subcase (ii): If $q\left(m_1'(t_1), t_2, v_1', v_2\right) = (0.5, 0.5)$, then agent 1's utility is $0.5\left(v_1\left(t_1\right) - v_2\left(t_2\right) + v_1\left(t_1 + t_2\right) - v_2\left(t_1 + t_2\right) + 2h\right)$. But we know

$$v_1\left(t_1\right) + v_2\left(t_1 + t_2\right) < v_2\left(t_2\right) + v_1\left(t_1 + t_2\right)$$

$$\Leftrightarrow$$

$$v_1\left(t_1 + t_2\right) - v_2\left(t_1 + t_2\right) + h > 0.5\left(v_1\left(t_1\right) - v_2\left(t_2\right) + v_1\left(t_1 + t_2\right) - v_2\left(t_1 + t_2\right) + 2h\right).$$

□

An interpretation of Proposition 4 is that given a two-stage mechanism, an untruthful ex-post Nash equilibrium exists. The untruthful ex-post Nash equilibrium is structured such that if an agent is allowed to obtain the resource first, he reports the negative of the value he would have received if he used the resource second. If an agent obtains the resource second, he reports the negative of the value he would have obtained if he used the resource first. Hence, it is assumed that agents can observe the ex-post types of others. Therefore, this equilibrium is only relevant when agents' private types become public information at some time. Another assumption of this equilibrium is that the manager believes that agents may have negative values. This is a relevant assumption when agents do not have a viable escape option. For example, imagine two cars stopped at an intersection that can only move forward or two entities bounded by a contract or constraint that forces them to play. In either of these two examples, agents may have value functions that are negative at specific times.

An interesting question to ask next is under what conditions do agents prefer one equilibrium over the other? More formally, which equilibrium results in higher utilities? I find that the answer depends on agents' value functions and types.

Proposition 5. *Let the direct mechanism q and T_i for all $i \in I$ be defined by eqs.* (7.1) *and* (7.2)*. For all $v_1, v_2 \in V$, $t_1, t_2 \in [0, 1]$,*

1. *$-v_2(t_2) > v_2(t_1+t_2)$ and $-v_1(t_1) > v_1(t_1+t_2)$ if and only if the ex-post Nash equilibrium defined by proposition 4 is strictly preferred to the truthful ex-post Nash equilibrium.*

2. *$-v_2(t_2) < v_2(t_1+t_2)$ and $-v_1(t_1) < v_1(t_1+t_2)$ if and only if the truthful ex-post Nash equilibrium is strictly preferred to the ex-post Nash equilibrium defined by proposition 4.*

Proof. Let arbitrary $v_1, v_2 \in V, t_1, t_2 \in [0,1]$. Consider item 1 and suppose $-v_2(t_2) > v_2(t_1+t_2)$ and $-v_1(t_1) > v_1(t_1+t_2)$.

Case 1. $q(t_1,t_2,v_1,v_2) = (1,0) \Leftrightarrow v_1(t_1) + v_2(t_1+t_2) > v_2(t_2) + v_1(t_1+t_2)$. Agent 1's and 2's utility in the untruthful ex-post Nash equilibrium is $v_1(t_1) - v_2(t_2) + h$ and $v_2(t_1+t_2) - v_1(t_1+t_2) + h$, respectively. Both agents' utilities in the truthful ex-post Nash equilibrium are $v_2(t_1+t_2) + v_1(t_1) + h$. Hence, we have

$$-v_2(t_2) > v_2(t_1+t_2)$$

$$\Leftrightarrow$$

$$v_1(t_1) - v_2(t_2) + h > v_2(t_1+t_2) + v_1(t_1) + h$$

and

$$-v_1(t_1) > v_1(t_1+t_2)$$

$$\Leftrightarrow$$

$$v_2(t_1+t_2) - v_1(t_1+t_2) + h > v_2(t_1+t_2) + v_1(t_1) + h$$

Case 2. Suppose it is the case that $v_1(t_1) + v_2(t_1+t_2) = v_2(t_2) + v_1(t_1+t_2)$. Agent 1's and 2's utility in the untruthful ex-post Nash equilibrium is $0.5(v_1(t_1) - v_2(t_2) + v_1(t_1+t_2) - v_2(t_1+t_2) + 2h)$ and $0.5(v_2(t_1+t_2) - v_1(t_1+t_2) + v_2(t_2) - v_1(t_1) + 2h)$, respectively. Both agents' utilities in the truthful ex-post

Nash equilibrium are $0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$. Hence, we have

$$-v_2\left(t_2\right)>v_2\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$0.5\left(v_1\left(t_1\right)-v_2\left(t_2\right)+v_1\left(t_1+t_2\right)-v_2\left(t_1+t_2\right)+2h\right)$$

$$>0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$$

and

$$-v_1\left(t_1\right)>v_1\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$0.5\left(v_2\left(t_1+t_2\right)-v_1\left(t_1+t_2\right)+v_2\left(t_2\right)-v_1\left(t_1\right)+2h\right)$$

$$>0.5\left(v_1\left(t_1\right)+v_2\left(t_1+t_2\right)+v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+2h\right)$$

Case 3. Suppose it is the case that $v_1\left(t_1\right)+v_2\left(t_1+t_2\right)<v_2\left(t_2\right)+v_1\left(t_1+t_2\right)$. Agent 1's and 2's utility in the untruthful ex-post Nash equilibrium is $v_1\left(t_1+t_2\right)-v_2\left(t_1+t_2\right)+h$ and $v_2\left(t_2\right)-v_1\left(t_1\right)+h$, respectively. Both agents' utilities in the truthful ex-post Nash equilibrium are $v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+h$. Hence, we have

$$-v_2\left(t_2\right)>v_2\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$v_1\left(t_1+t_2\right)-v_2\left(t_1+t_2\right)+h>v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+h$$

and

$$-v_1\left(t_1\right)>v_1\left(t_1+t_2\right)$$

$$\Leftrightarrow$$

$$v_2\left(t_2\right)-v_1\left(t_1\right)+h>v_2\left(t_2\right)+v_1\left(t_1+t_2\right)+h$$

By changing the inequalities, item 2 can also be shown to be true, and the backward equivalent statements can show the reverse implication. □

One clear implication from proposition 5 is that if agents themselves are willing to pay to use the resource at any time and know that the other agent is willing to pay to use the resource at any time (i.e., mathematically, this means that both agents' value functions are positive), then it is in the best interest of both agents to report truthfully in the second stage. However, suppose both agents are only willing to receive money, and each agent knows that the other agent is also only willing to receive money. In that case, agents may implicitly collude with one another to increase individual utilities. A plausible situation where agents may have negative valuations may be when agents have no escape option (e.g., they are stuck with perishable goods at the intersection).

While I have only compared one untruthful ex-post Nash equilibrium with the truthful ex-post Nash equilibrium, I will ignore other potentially untruthful ex-post Nash equilibria in this paper. However, in general, proposition 5 provides a framework for finding the conditions of agents' times and value functions such that agents have a preference over equilibria. An ex-post Nash equilibrium that I will state briefly and which is somewhat important for determining whether the two-stage mechanism should be implemented in reality is as follows.

Corollary 3. *Let the direct mechanism q and T_i for all $i \in I$ be defined by eqs. (7.1) and (7.2). If agents' value functions are constant, then reporting truthful times and value functions $v_i(t) = 1$ for all $t \in [0, 2]$ and $i \in I$ in the first stage and both agents reporting a payoff utility of 1 in the second stage is an ex-post Nash equilibrium in the direct mechanism.*

Proof. This is simply a variation of proposition 4. □

While the case of constant value functions is not the primary focus of this paper, it is important to note that if agents' value functions are constant (i.e., agents value using the resource earlier the same as using the resource later), then the two-stage mechanism will likely encounter untruthful reports. Untruthful reporting in the second stage, such as in corollary 3, may result in the manager accumulating a severe budget deficit.

9 Model with Manager Not Knowing Agents' Values & Types Over Multiple Rounds

In this section, I construct a model to explore the variation in the manager's budget given the ex-ante budget balanced mechanism proposed in section 7. I begin by defining the assumptions of the model.

Let $R \in \mathbb{N}$ represent the total number of rounds that the manager must interact with two agents. A round ends when both agents have used the resource, and the next round begins when two new agents desire to use the resource. For each round $r \in \mathbb{N}_R = \{1, \ldots, R\}$, let the expected times it takes for agents 1 and 2 to use the resource be $\boldsymbol{t}^r := (t_1^r, t_2^r) \in [0,1]^2$. For any round $r \in \mathbb{N}_R$, let the probability measure of $\boldsymbol{t}^r$ be the Lebesgue measure $\mu : \mathcal{B}\left([0,1]^2\right) \to [0,1]$ such that $\mu\left([0,1]^2\right) = 1$. Assume for any two rounds $r_1, r_2 \in \mathbb{N}_R, \boldsymbol{t}^{r_1}$ and $\boldsymbol{t}^{r_2}$ are independent. Together, these imply that the real-valued random variable $\boldsymbol{t}^r$ are independently uniformly distributed. Lastly, assume that the distribution of the random variables remains the same in each round.

For each round $r \in \mathbb{N}_R$, let $V := \left\{-t+1, -\frac{1}{2}t + \frac{1}{2}\right\}$, and let agents' valuation functions be $\boldsymbol{v}^r := (v_1^r, v_2^r) \in V^2$ such that each function in V has an equal probability of being chosen. Thus, for each round $r \in \mathbb{N}_R$, let the probability measure of $\boldsymbol{v}^r$ be $\eta : \mathcal{P}\left(V^2\right) \to [0,1]$ such that $\eta\left(V^2\right) = 1$, for all $X \in V^2, \eta(\{X\}) = \frac{1}{|V^2|}$, and $\mathcal{P}(X)$ is the power set of X. Assume for any two rounds $r_1, r_2 \in \mathbb{N}_R, \boldsymbol{v}^{r_1}$ and $\boldsymbol{v}^{r_2}$ are independent, and assume the probability distribution of the random variables remains the same in each round. Together, these imply that the function-valued random variable $\boldsymbol{v}^r$ is independently uniformly distributed. Lastly, assume that the distribution of the random variables remains the same in each round. While these are the specific distributions that I analyze in this model, they could theoretically be substituted for other distributions, and the analysis resulting from such a substitution would parallel mine.

To analyze ex-ante budget balanced, further suppose that each agent reports their value function and types truthfully in every round. Let the additional charge h be defined by eq. (7.3). This implies that in each round, the additional charge does not change. For each round $r \in \mathbb{N}_R$, let

$$\begin{aligned} T_1^r &:= h + \begin{cases} v_2^r\left(t_1^r + t_2^r\right) & \text{if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) > v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \\ v_2^r\left(t_2^r\right) & \text{if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) < v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \end{cases} \\ T_2^r &:= h + \begin{cases} v_1^r\left(t_1^r\right) & \text{if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) > v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \\ v_1^r\left(t_1^r + t_2^r\right) & \text{if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) < v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \end{cases} \end{aligned} \tag{9.1}$$

represent the transfers agents one and two make or receive at round r.

I now describe the notation necessary to analyze the variation of the manager's budget over some number of rounds. For all $r \in \mathbb{N}_R$, let $B_r := \sum_{i=1}^{r} T_1^i + T_2^i$ represent the aggregate transfers or the manager's budget up to round r, and for technicality let $B_0 = 0$.

Definition 8. *Suppose that* $\{X_n\}_{n\geq 1}$ *is a sequence of* $\mathbb{R}$*-valued independent and identically distributed random variables. Let* $\{S_n\}_{n\geq 0}$ *be a sequence such that* $S_0 \in \mathbb{R}$ *and for all* $n \geq 1$*,*

$$S_n = S_{n-1} + X_n$$

A random walk is a pair $\left(\{S_n\}_{n\geq 0}, \{X_n\}_{n\geq 1}\right)$*.*

10 Analysis of the Ex-Ante Budget Balanced Mechanism Over Multiple Rounds

The purpose of this section is to understand the implications of the mechanism proposed in section 7 by considering its performance across many rounds. Specifically, when agents report truthfully in each round, does the manager's budget remain near zero after participating in the mechanism for several rounds? Results from probability theory illustrate significant variation in the manager's budget as the number of rounds increases.

Lemma 1. *If* $h = -\frac{43}{216}$*, then for all* $r \in \mathbb{N}_R$ *and agent* $i \in I, E\left[T_i^r\right] = 0$*.*

Proof. Let $h = -\frac{43}{216}$, let arbitrary $r \in \mathbb{N}_R$, and, WLOG, consider agent 1. By eq. (9.1), we know

$$T_1^r := h + \begin{cases} v_2^r\left(t_1^r + t_2^r\right) \text{ if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) > v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \\ v_2^r\left(t_2^r\right) \text{ if } v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) < v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right) \end{cases}.$$

By how we defined our model, we also know the probability measures η and μ. Let the sets A and B be defined by

$$A := \left\{\left(t_1^r, t_2^r\right) \in [0,1]^2 \mid v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) > v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right)\right\} \tag{10.1}$$

and

$$B := \left\{\left(t_1^r, t_2^r\right) \in [0,1]^2 \mid v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right) < v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right)\right\}. \tag{10.2}$$

Then we have

$$E\left[T_1^r\right] = -\frac{43}{216} + \sum_{(v_1^r, v_2^r) \in V^2} \eta\left(v_1^r, v_2^r\right) \int_{[0,1]^2} v_2^r\left(t_1^r + t_2^r\right) 1_A + v_2^r\left(t_2^r\right) 1_B d\mu = 0$$

□

Lemma 1 shows that for each round $r \in \mathbb{N}_R$, each agent must pay an additional charge of $\frac{43}{216}$ in order for the manager to maintain an ex-ante balanced budget. Hence, for the remaining of the section let $h = -\frac{43}{216}$.

Lemma 2. *For each round* $r \in \mathbb{N}_R, \operatorname{Var}\left(T_1^r + T_2^r\right) = \frac{545}{2304}$.

Proof. From lemma 1 and the fact that $h = -\frac{43}{216}$ we know $E\left[T_1^i + T_2^i\right] = E\left[T_1^i\right] + E\left[T_2^i\right] = 0$ for all $i \in \mathbb{N}_R$. By eq. (9.1), we know that the transfer of each agent depends on their value function and type. Further, we know the probability measures η and μ. Let the sets A and B be defined by eqs. (10.1) and (10.2). Then we have for any $r \in \mathbb{N}_R$,

$$\begin{aligned}
&\operatorname{Var}\left(T_1^r + T_2^r\right) = E\left[\left(T_1^r + T_2^r - E\left[T_1^r + T_2^r\right]\right)^2\right] \\
&= E\left[\left(T_1^r + T_2^r\right)^2\right] \\
&= \sum_{(v_1^r, v_2^r) \in V^2} \eta\left(v_1^r, v_2^r\right) \int_{[0,1]^2} \left(2h + \left(v_1^r\left(t_1^r\right) + v_2^r\left(t_1^r + t_2^r\right)\right) 1_A + \left(v_2^r\left(t_2^r\right) + v_1^r\left(t_1^r + t_2^r\right)\right) 1_B\right)^2 d\mu \\
&= \frac{545}{2304}
\end{aligned}$$

□

Lemma 2 will be useful in the next proposition.

Proposition 6. *Consider the manager's budget as a random walk* $\left(\{B_n\}_{n \geq 0}, \{T_1^r + T_2^r\}_{r \geq 1}\right)$ *such that* $B_0 = 0$ *and for all* $n \geq 1$,

$$B_n = \sum_{i=1}^{n} T_1^i + T_2^i.$$

As $n \to \infty, B_n$ *has an asymptotically normal distribution with mean zero and variance* $\frac{545}{2304} n$.

Proof. Since $\left\{T_1^i + T_2^i\right\}_{i \in \mathbb{N}_R}$ is a sequence of independent and identically distributed random variables for any R such that $E\left[T_1^i + T_2^i\right] = E\left[T_1^i\right] + E\left[T_2^i\right] = 0$ for all $i \in \mathbb{N}_R$ by lemma 1, by the central limit theorem, as

$n \to \infty$,

$$z_n := \frac{B_n - nE\left[T_1^1 + T_2^1\right]}{\sqrt{n}} = \frac{B_n}{\sqrt{n}}$$

converges to a normal distribution $\mathcal{N}\left(0, E\left[\left(T_1^1 + T_2^1\right)^2\right]\right)$ where $E\left[\left(T_1^1 + T_2^1\right)^2\right] = \frac{545}{2304}$ by lemma 2. Hence, as $n \to \infty, B_n = \sqrt{n}z_n$ converges to a normal distribution $\mathcal{N}\left(0, \frac{545}{2304}n\right)$. □

In essence, proposition 6 states that the manager's budget will be approximately normally distributed with a variance proportional to the number of rounds. Hence, it is likely that the manager runs a deficit or surplus after many rounds. Therefore, the approximate probability of the manager's budget being bounded between some number can be derived from the normal distribution.

11 Discussion of Results & Extensions of Models

There are many ways in which the models I have proposed can be easily extended, but there are also many situations in which it is difficult to implement the mechanisms I propose. In this section, I elaborate upon both situations that my models cover and situations in which the mechanisms I propose should not be used.

In some situations, the agent may not know their times because of external factors, and the two-stage mechanism described in section 7 should only be used on a case-by-case basis. For example, perhaps an agent wants to submit a task to a computer cluster, and the agent does not have any information on the speed at which the computer cluster operates. Suppose that only the manager knows that the time it takes for each task to complete is the same, say t_0. However, he does not know the values agents place on completing their tasks at various times. In such a situation, the manager may ask agents to report their values for completing their tasks at t_0 and $2t_0$ times in the first stage, and he may still use the two-stage mechanism. This example also illustrates that in many situations, what is private information to the manager is agents' valuations and not their times. Under certain conditions on agents' value functions, proposition 5 shows that the second or payment stage in the two-stage mechanism extracts agents' valuations. Hence, the purpose of transfers is to incentivize agents to truthfully report private information that the manager does not know.

In some cases, agents do not know their times because of a lack of experience. Suppose it is the first time that an agent must report the time he will take to use the resource. In such situations, assuming the agent does not know his expected time to use the resource, the two-stage mechanism should not be used altogether.

In the models I introduced in sections 3 and 6, I assumed that agents' times are drawn from a closed interval $[0, 1]$. In some situations, agents' times may be drawn from some finite set. Perhaps this finite set contains the number of hours or days agents expect to take to use the resource. If all other aspects of the models described by sections 3 and 6 are true, then the properties of their respective mechanisms are the same with slight changes to the payment constant (i.e., the manager must know the probability distribution over some discrete sample space instead of a continuous sample space).

Another important aspect of the model described in section 6 is agents' value functions. For the two-stage mechanism to be employed, the manager must know the possible value functions of agents and the probability over the possible value functions. Next, to avoid agents from explicitly colluding with the goal of extracting as much money from the manager as possible, agents' value functions must not be constant. In other words, agents must have different values for using the resource at different times. If agents' value functions satisfy these desirable properties, the manager must also consider the change in his budget across several rounds. Proposition 6 shows that the manager's budget will inevitably run a surplus or deficit after many rounds but also provides a framework for analyzing the bounds of the manager's budget given some probability.

12 Conclusion & Future Research

In this paper, I have focused on investigating collusive behavior and the change in the manager's budget when agents' values are private and interdependent. For the one-stage mechanism when the manager has almost perfect information, described in section 4, collusion is unlikely because agents will obtain the same utility when colluding as when reporting truthfully, shown in section 5. For the two-stage mechanism, there are situations in which it is beneficial for agents to implicitly or explicitly collude. Section 8 describes non-truthful equilibria agents may prefer over reporting truthfully in the two-stage mechanism. When agents do report truthfully, and the manager uses the two-stage mechanism over multiple rounds described in section 9, the manager will inevitably run a surplus or deficit shown in section 10. Further research may investigate whether an efficient mechanism that limits the bounds of the manager's budget exists and its actual format.

Various other paths of research may begin from where I have left off. So far, the two models I have proposed assume that two agents desire to use the resource at a specific time. Extending the proposed models may allow agents to join the mechanism after some time. For a concrete example to motivate such a model, consider the situation when two trucks need to cross a country's border, and each truck needs to be inspected in order to cross. One truck may arrive at time x_0 and may need t_0 time to be checked. Another truck may arrive at time $x_0 + \epsilon$ where $\epsilon > 0$ is some small number and may need t_1 time to be checked. Future research may investigate budget balanced mechanisms that maximize efficiency depending on the values of ϵ, t_0, and t_1. The research may start with perfect information, in that the manager knows such a schedule of information. Such research would have vast applications in determining the order in which agents are allowed to borrow resources.

Another path of research extending from the model in section 3 may investigate the general format of mechanisms that are ex-post incentive compatible and ex-post budget balanced. While I only focus on analyzing one such mechanism, other ex-post incentive compatible and ex-post budget balanced mechanisms exist. Furthermore, for the two-stage mechanism, since I only investigate a particular untruthful ex-post Nash equilibrium, further research may investigate the general format of untruthful ex-post Nash equilibria if such a format exists and investigate the change in the manager's budget when a mix of these untruthful equilibria and the truthful equilibrium is reached across several rounds. A harder but more interesting research topic along these lines would be to investigate whether there are mechanisms that result in truthfully reporting as a unique equilibrium (where the definition of equilibrium would need to be defined). The goal of such research would be to resolve the issue of collusion entirely.

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